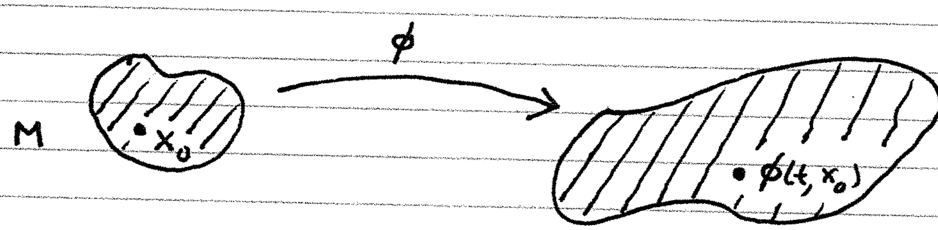
Positive Invariance and Trapping Regions

Recall the flow function $\phi(t, x_0)$ for $\dot{x} = f(x)$ satisfies

$$(1) \qquad \frac{\partial \phi}{\partial t} = f(\phi)$$

$$(2) \qquad \phi(0, x_0) = x_0$$

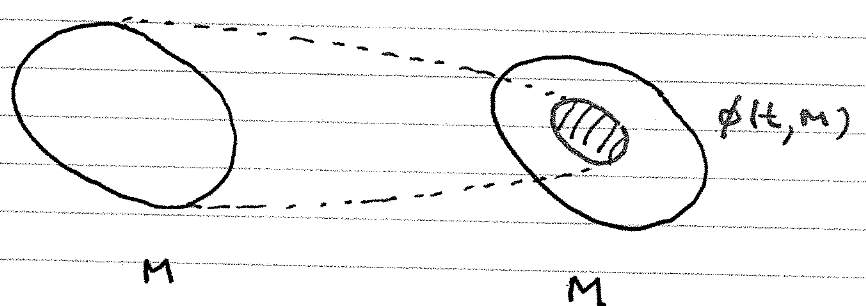
One can examine the image of sets M under the flow ϕ



Definition A set M is positively invariant if

$$\phi(t, M) \subset M \qquad \forall t \geq 0$$

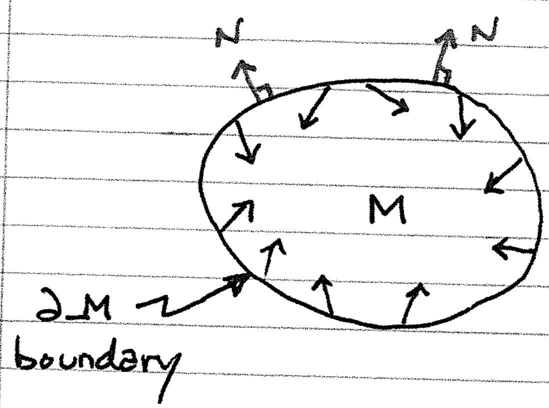
If additionally it si closed and bounded then M is called a trapping region



One way to check if a set M is a trapping region is to show

$$f \cdot N < 0 \qquad \forall x \in \partial M$$

where N is the outward normal on the boundary ∂M of the region M.



Illustrates a case where f is always pointing into the interior of M and

$$f \cdot N < 0 \qquad \forall x \in \partial M$$

EXAMPLE
$$\dot{x} = -xy - x$$
$$\dot{y} = -y$$

∂M is the unit circle and M its interior.

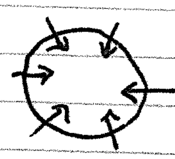
∂M is parametrized by $(x,y) = (\cos\theta, \sin\theta)$ and $N = (\cos\theta, \sin\theta)$ as well.

$$F(\theta) = -(xy + x, y) \cdot N$$

algebra

$$F(\theta) = -(1 + \sin\theta \cos^2\theta) < 0$$

Hence M is a trapping region.



Note M contains one fixed pt

Poincaré - Bendixson Theorem

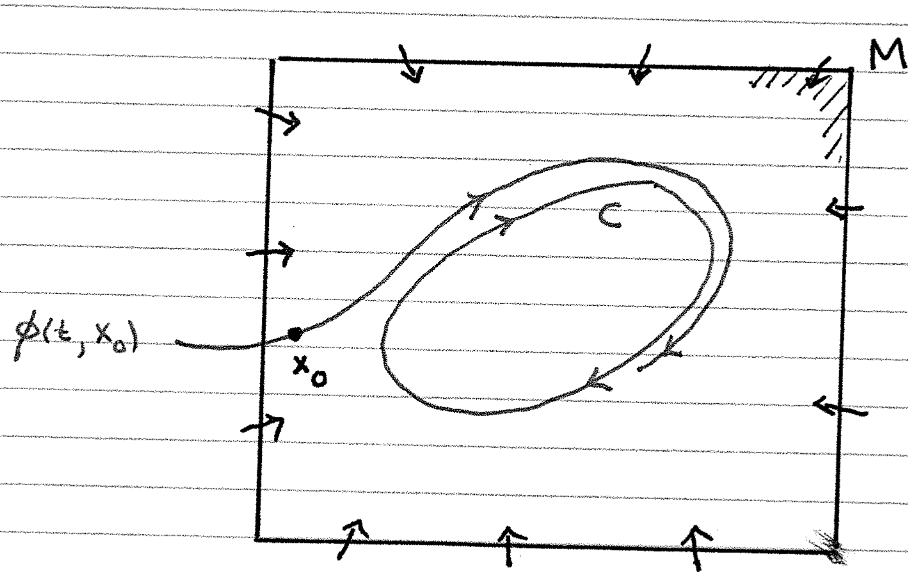
Suppose M is a trapping region for $\dot{x} = f(x)$ which contains no fixed points. Then there exists some periodic orbit C contained entirely in M.

Pf: Difficult. See Perko (1991: pg 227) for details. May be many C in M.

Remarks

(1) There are more general versions (later) where M may contain a finite number of fixed points. The conclusion in such generalizations are more complex.

(2) Not only does C exist, but $\phi(t, x_0)$ "approaches" C as $t \to \infty$. "Approach" needs a more careful definition

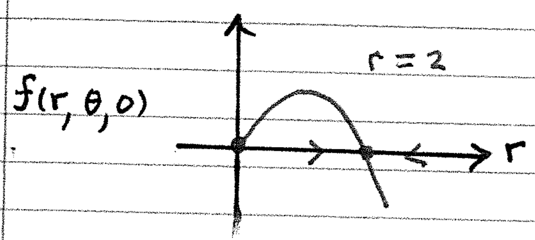


C = periodic orbit

EXAMPLE Annular trapping region

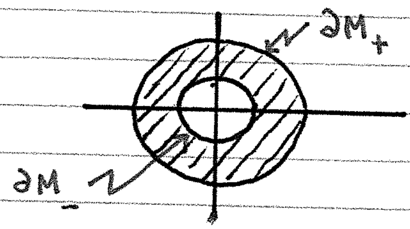
$$\dot{r} = r(4-r^2) + \mu r \sin\theta = f(r,\theta,\mu) \qquad (1)$$
$$\dot{\theta} = 1 \qquad (2)$$

when $\mu = 0$ this system has a stable periodic orbit.



Does the orbit persist for $\mu \neq 0$?

Annular trapping region M. origin sole fx pt.



Want (for trapping)

$\dot{r} > 0$ on ∂M_-
$\dot{r} < 0$ on ∂M_+

On ∂M_- we need a radius r small enough so $f > 0$

$$r(4-r^2) > -\mu r \sin\theta$$

Sufficient to choose r so that $r(4-r^2) > +\mu r \geq -\mu r \sin\theta$

$$\overbrace{r(4-\mu-r^2)}^{\text{want } +} > 0$$

True if $r > \sqrt{4-\mu}$ on inner circle ∂M_-.

Similarly $r < \sqrt{4+\mu}$ on outer circle ∂M_+ so $f < 0$
Thus if these radii are satisfied M is indeed a trapping region. By Poincaré–Bendixson there is at least one periodic orbit C inside M.

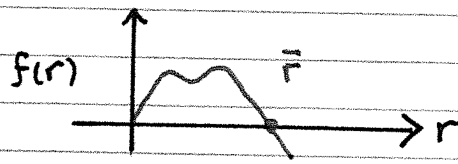
EXAMPLE Polar generalizations

$$\dot{r} = f(r) + \mu G(r,\theta)$$
$$\dot{\theta} = 1$$

where G is uniformly bounded

$$|G(r,\theta)| < K \qquad \forall (r,\theta)$$

and the graph of $f(r)$ qualitatively is

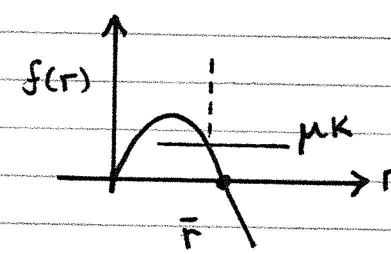


Same annular M. On inner boundary ∂M_- need

$$f(r) > -\mu G(r,\theta)$$

Sufficient that we pick r such that

$$f(r) > \mu K$$



which is true for suff. small μ
Similar for ∂M_+ so M is a trapping region.

EXAMPLE
$$\dot{r} = r(1-r) + \frac{\mu r^2 \sin^2\theta}{1+r^2}$$
$$\dot{\theta} = 1$$

Bounded so we can find an annular trapping region and system has period orbit.

Omega limit sets

Below $\phi(t, x_0)$ is the flow function for

$$\dot{x} = f(x) \qquad x(0) = x_0$$

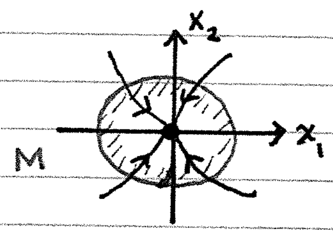
Defn: p is an ω-limit point of x_0 if there is a sequence of times $\{t_i\}$ with $t_i \to \infty$ such that

$$\phi(t_i, x_0) \longrightarrow p \qquad \text{as } i \to \infty$$

The ω-limit set $\omega(x_0)$ of x_0 is

$$\omega(x_0) \equiv \{p : p \text{ an } \omega\text{-limit point}\}$$

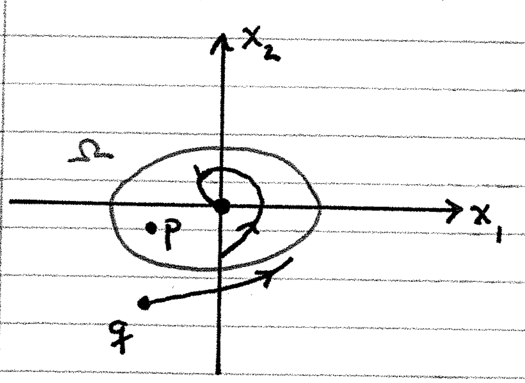
EXAMPLE Stable node



M is positively invariant

$$\omega(x_0) = \{(0,0)\} \qquad \forall x_0 \in M$$

EXAMPLE Half stable periodic orbit

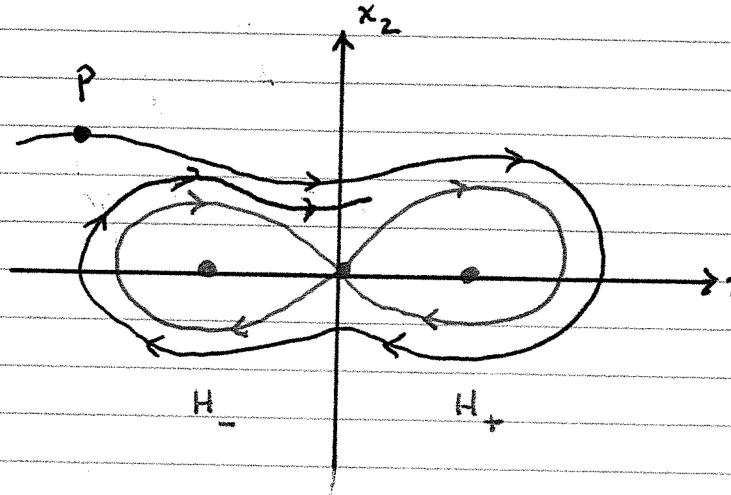


Limit cycle Ω, stable origin

$$\omega(p) = \vec{0}$$

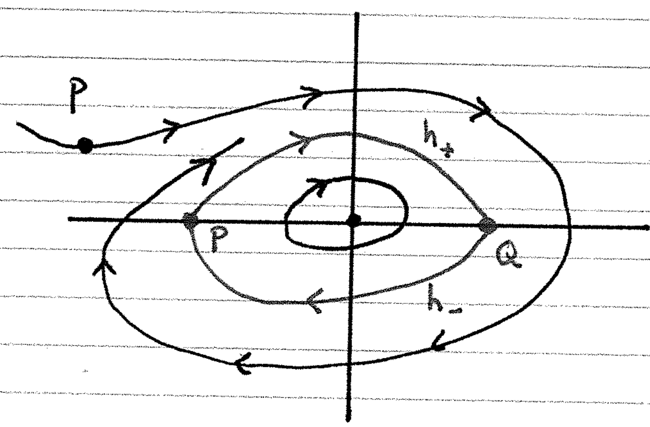
$$\omega(q) = \Omega$$

EXAMPLE Homoclinic orbit(s)



$\omega(p)$ is the union of the origin and two homoclinic orbits $H_\pm$

EXAMPLE Heteroclinic cycle



$\omega(p)$ is the union of the two heteroclinic obits $h_\pm$ and fixed pts P and Q

Poincare - Bendixson (General)

Suppose M is a trapping region for

$$\dot{x} = f(x) \qquad x(0) = x_0$$

which contains a finite number of isolated fixed points $\bar{x}_i$, $i = 1, 2 .. n$. Then for each $x_0 \in M$ one of the following is true:

(a) $\omega(x_0)$ is a fixed point

(b) $\omega(x_0)$ is a periodic orbit in M

(c) $\omega(x_0)$ consists of a finite number of fixed points, homoclinic orbits and/or heteroclinic orbits.

Remark: The theorem implies that planar systems can't have "chaos"! At most

(i) fixed points

(ii) periodic orbits

(iii) homoclinic cycles

(iv) heteroclinic cycles