Positive Invariance and Trapping Regions

Recall the flow function $\varphi(t, x_0)$ for $\dot{x} = f(x)$ satisfies

(1) \[ \frac{\partial \varphi}{\partial t} = f(\varphi) \]
(2) \[ \varphi(0, x_0) = x_0 \]

One can examine the image of sets $M$ under the flow $\varphi$

Definition A set $M$ is positively invariant if

$\varphi(t, M) \subseteq M \quad \forall t \geq 0$

If additionally it is closed and bounded then $M$ is called a trapping region.
One way to check if a set \( M \) is a trapping region is to show

\[
f \cdot N < 0 \quad \forall x \in \partial M
\]

where \( N \) is the outward normal on the boundary \( \partial M \) of the region \( M \).

Illustrates a case where \( f \) is always pointing into the interior of \( M \) and

\[
f \cdot N < 0 \quad \forall x \in \partial M
\]

**Example**

\[
x = -xy - x \\
y = -y
\]

\( \partial M \) is the unit circle and \( M \) its interior.

\( \partial M \) is parametrized by \((x, y) = (\cos \theta, \sin \theta)\) and \( N = (\cos \theta, \sin \theta) \) as well.

\[
F(\theta) = -(xy + x, y) \cdot N
\]

\[
\text{algebra}\quad F(\theta) = -(1 + \sin \theta \cos^2 \theta) < 0
\]

Hence \( M \) is a trapping region.

Note \( M \) contains one fixed pt.
Poincaré - Bendixson Theorem

Suppose $M$ is a trapping region for $\dot{x} = f(x)$ which contains no fixed points. Then there exists some periodic orbit $C$ contained entirely in $M$.

**Pf:** Difficult. See Perko (1991: pg 227) for details. May be many $C$ in $M$.

**Remarks**

1. There are more general versions (later) where $M$ may contain a finite number of fixed points. The conclusion in such generalizations are more complex.

2. Not only does $C$ exist, but $\phi(t, x_0)$ "approaches" $C$ as $t \to \infty$. "Approach" needs a more careful definition.
EXAMPLE

Annular trapping region

\[ r = r(4-r^2) + \mu r \sin \theta = f(r, \theta, \mu) \]

\[ \dot{\theta} = 1 \]

when \( \mu = 0 \) this system has a stable periodic orbit.

\[ f(r, \theta, 0) \]

\[ r = 2 \]

Does the orbit persist for \( \mu \neq 0 \)?

Annular trapping region \( M \). Origin solely \( f \times p^+ \).

\[ \partial M_+ \]

Want (for trapping)

\[ \begin{align*}
  r &> 0 \quad \text{on } \partial M_- \\
  \dot{r} &< 0 \quad \text{on } \partial M_+ 
\end{align*} \]

On \( \partial M_- \) we need a radius \( r \) small enough so \( f > 0 \)

\[ r(4-r^2) > -\mu r \sin \theta \]

Sufficient to choose \( r \) so that \( r(4-r^2) + \mu r \geq -\mu r \sin \theta \)

\[ r(4-\mu+r^2) > 0 \]

True if \( r > \sqrt{4-\mu} \) on inner circle \( \partial M_- \).

Similarly \( r < \sqrt{4+\mu} \) on outer circle \( \partial M_+ \) so \( f < 0 \)

Thus if these radii are satisfied \( M \) is indeed a trapping region. By Poincaré-Bendixson there is at least one periodic orbit \( C \) inside \( M \).
**EXAMPLE** Polar generalizations

\[
\begin{align*}
\dot{r} &= f(r) + \mu G(r, \theta) \\
\dot{\theta} &= 1
\end{align*}
\]

where \( G \) is uniformly bounded

\[ |G(r, \theta)| < K \quad \forall (r, \theta) \]

and the graph of \( f(r) \) qualitatively is

\[ f(r) \]

Same annular \( M \). On inner boundary \( \partial M \), need

\[ f(r) > -\mu G(r, \theta) \]

Sufficient that we pick \( r \) such that

\[ f(r) > \mu K \]

which is true for suff. small \( \mu \).

Similar for \( \partial M_+ \), so \( M \) is a trapping region.

**EXAMPLE**

\[
\begin{align*}
\dot{r} &= r(1-r) + \frac{\mu r^2 \sin^2 \theta}{1 + r^2} \\
\dot{\theta} &= 1
\end{align*}
\]

Bounded so we can find an annular trapping region and system has period orbit.
Omega limit sets

Below $\phi(t, x_0)$ is the flow function for

$$\dot{x} = f(x) \quad x(0) = x_0$$

**Defn:** $p$ is an $\omega$-limit point of $x_0$ if there is a sequence of times $\{t_i\}$ with $t_i \to \infty$ such that

$$\phi(t_i, x_0) \to p \quad \text{as} \quad i \to \infty$$

The $\omega$-limit set $\omega(x_0)$ of $x_0$ is

$$\omega(x_0) = \{ p : p \text{ is an } \omega\text{-limit point} \}$$

**Example** Stable node

M is positively invariant

$$\omega(x_0) = \{(0,0)\} \quad \forall x_0 \in M$$

**Example** Half stable periodic orbit

Limit cycle $\Omega$, stable origin

$$\omega(p) = \partial$$

$$\omega(q) = \Omega$$
**Example** Homoclinic orbit(s)

\[ w(p) \text{ is the union of the origin and two homoclinic orbits } H_\pm \]

**Example** Heteroclinic cycle

\[ w(p) \text{ is the union of the two heteroclinic orbits } h_+ \text{ and fixed pts } P \text{ and } Q \]
Poincaré–Bendixson (General)

Suppose $M$ is a trapping region for

$$\dot{x} = f(x), \quad x(0) = x_0$$

which contains a finite number of isolated fixed points $\bar{x}_i, i = 1, 2, \ldots, n$.

Then for each $x_0 \in M$ one of the following is true:

(a) $\omega(x_0)$ is a fixed point

(b) $\omega(x_0)$ is a periodic orbit in $M$

(c) $\omega(x_0)$ consists of a finite number of fixed points, homoclinic orbits and/or heteroclinic orbits.

Remark: The theorem implies that planar systems can't have "chaos." At most

(i) fixed points

(ii) periodic orbits

(iii) homoclinic cycles

(iv) heteroclinic cycles