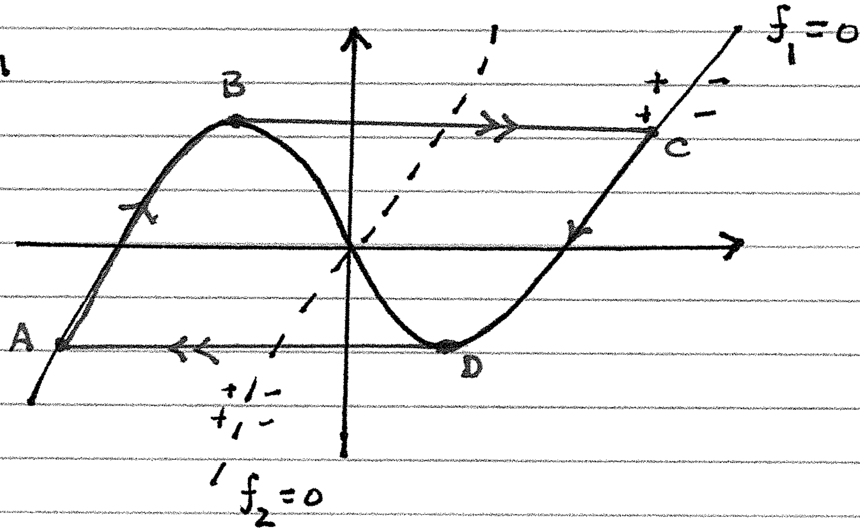
Relaxation oscillators

Is a periodic orbit where one dependent variable varies much quicker than the other. Eqns are

$$(1) \qquad \dot{x} = f_1(x,y)$$
$$(2) \qquad \dot{y} = \varepsilon f_2(x,y) \qquad 0 < \varepsilon \ll 1$$

where ε is a small positive number. Qualitatively the nullclines have the assumed shapes

FIG 1

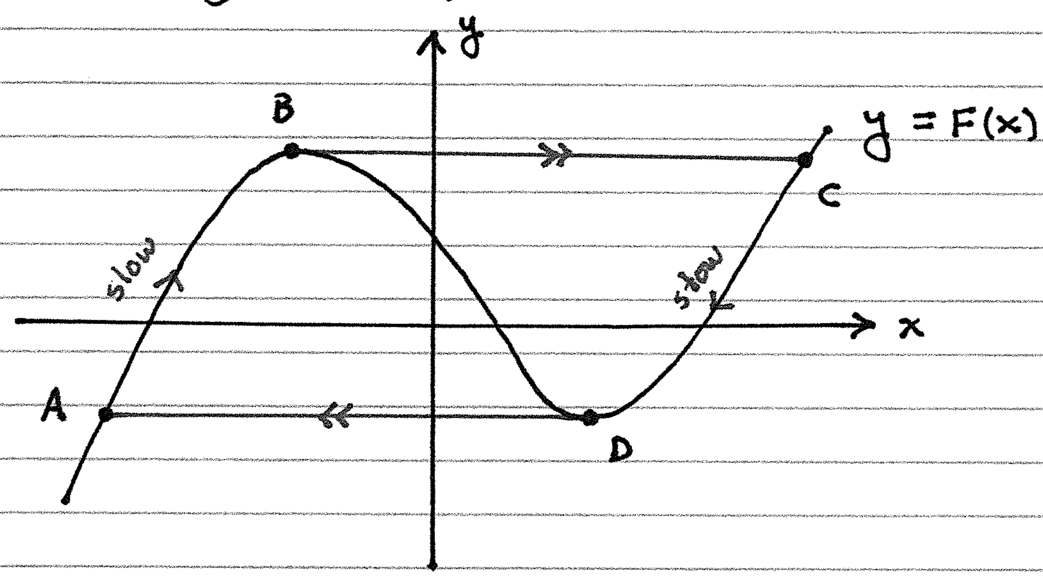


x = fast
y = slow

Can prove (Poincare) periodic orbit near ABCD exists and that its period T is

$$T \simeq T_{AB} + T_{CD}$$

Computing the approximate period



We assume that on the x-nullcline $f_1(x,y)=0$ implies $y = F(x)$ for some function $F(x)$.

We use the notational convention

$$\alpha = (x_\alpha, y_\alpha) \qquad \alpha = A, B, C, D$$

Introduce a slow time τ

$$\tau = \varepsilon t$$

So the differential equations are

$$\boxed{\begin{aligned} &(3) \qquad \varepsilon \frac{dx}{d\tau} = f_1(x,y) \\ &(4) \qquad \frac{dy}{d\tau} = f_2(x,y) \end{aligned}}$$

Note that when $\varepsilon = 0$ in (3) we have

$$f_1(x,y) = 0$$

or that the solution lies on the $f_1 = 0$ nullcline as depicted in the figure along AB and CD. BC and DA are rapid transitions. Of course in reality ε is only small hence trajectories are just near that nullcline.

Seek a leading order approximation by the expansions

$$x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + \cdots$$

$$y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + \cdots$$

The leading equations are:

$$\left.\begin{array}{ll} (5) & 0 = f_1(x_0, y_0) \\ (6) & \dfrac{dy_0}{d\tau} = f_2(x_0, y_0) \end{array}\right\} \text{"leading order slow subsystem"}$$

But since (5) $\Rightarrow y_0 = F(x_0)$ so

$$(7) \qquad \frac{dy_0}{d\tau} = H(x_0) \equiv f_2(x_0, F(x_0))$$

But again since $y_0 = F(x_0)$ we have

$$\frac{dy_0}{d\tau} = F'(x_0)\,\frac{dx_0}{d\tau}$$

Using this in (7) we obtain a decoupled differential equation for x_0

$$\boxed{\frac{dx_0}{d\tau} = \frac{H(x_0)}{F'(x_0)}} \qquad (8)$$

So, from this we deduce

$$T_{AB} = \int_{x_A}^{x_B} \frac{F'(x)}{H(x)}\,dx$$

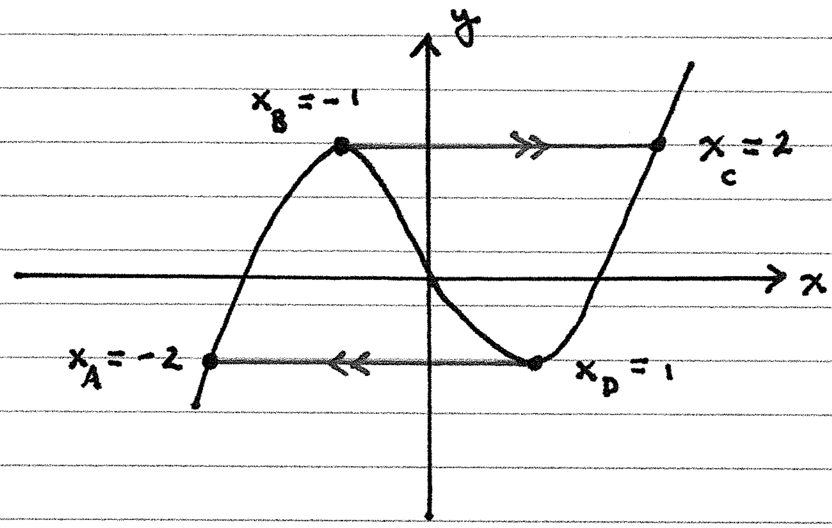
and a similar expression for T_{CD} $\Rightarrow$

$$T \simeq \int_{x_A}^{x_B} \frac{F'(x)}{H(x)}\,dx + \int_{x_C}^{x_D} \frac{F'(x)}{H(x)}\,dx$$

EXAMPLE

$$\dot{x} = y - F(x)$$
$$\dot{y} = -\varepsilon x$$

$F(x) \equiv \frac{1}{3}x^3 - x$



$F'(x) = 0$ at $x = \pm 1$ hence x_B, x_D values

Leading order slow subsystem $\tau = \varepsilon t$

$$(1) \qquad y = F(x)$$
$$(2) \qquad y' = -x$$

from which we find $y' = F'(x)\,x'$ $\Rightarrow$

$$\frac{dx}{d\tau} = -\frac{x}{F'(x)}$$

$F'(x) = x^2 - 1$

Separate and integrate

$$T_{AB} = \int_{-2}^{-1} \frac{-F'(x)}{x}\,dx = 3 - 2\ln 2$$

By symmetry $T_{CD} = T_{AB}$ so total period is

$$T \simeq 2T_{AB} = 6 - 4\ln 2$$