Bifurcations - overview

$$\dot{x} = f(x, \mu) \qquad x \in \mathbb{R}^2 \qquad \mu \in \mathbb{R}^m$$

Typically $m = 1, 2$ for the parameter μ.
As μ is varied the stability and number of the following may change.

(i) Equilibria / fixed point

(ii) Periodic orbits

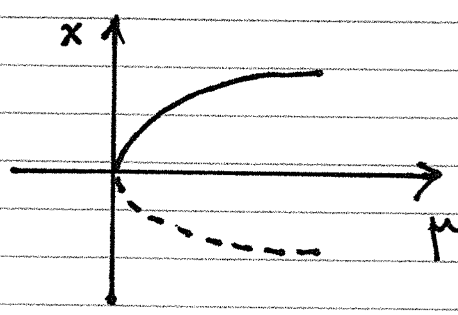
(iii) Hetero and homoclinic orbits

When $\mu \in \mathbb{R}$ it is common to summarize such changes in a bifurcation diagram. To make a 2D-diagram one plots a measure of x versus μ, such as

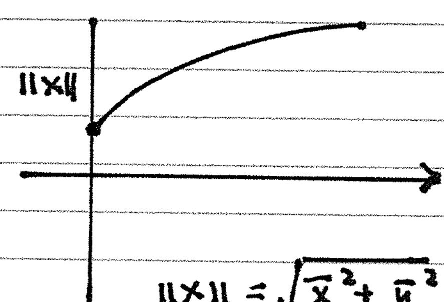
$$x_k\text{- component}, \quad \|x\|\text{-norm}, \quad \max_t x_k(t)$$

For instance

$$\dot{x} = \mu - x^2 \qquad \bar{x}(\mu) = \pm\sqrt{\mu}$$
$$\dot{y} = 1 - y \qquad \bar{y}(\mu) = 1$$



Component



$$\|x\| = \sqrt{\bar{x}^2 + \bar{y}^2}$$

norm

Bifurcation of Equilibria of $\dot{x} = f(x,\mu)$, $\mu \in \mathbb{R}$

A branch $\bar{x}(\mu)$ of fixed points is a function that satisfies

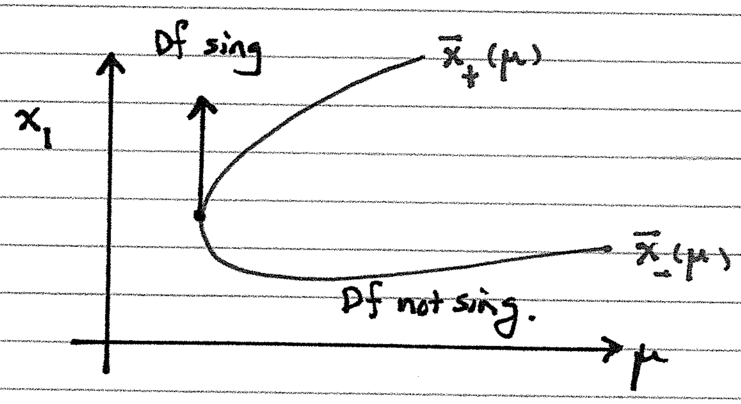
$$(1) \qquad f(\bar{x}(\mu), \mu) = 0$$

$\forall \mu \in I$ (some interval). There may be many different branches on the same I. Differentiate (1) in μ

$$(2) \qquad Df(\bar{x}, \mu)\frac{d\bar{x}}{d\mu} + f_\mu(\bar{x}, \mu) = 0$$

This implies a branch persists (IFT) so long as the Jacobian $Df(\bar{x}, \mu)$ nonsingular. In this case $\bar{x}'(\mu)$ is well defined

$$\frac{d\bar{x}}{d\mu} = -Df(\bar{x}, \mu)^{-1} f_\mu(\bar{x}, \mu)$$



shows two branches collide when $\bar{x}'(\mu)$ not define. Df singular!

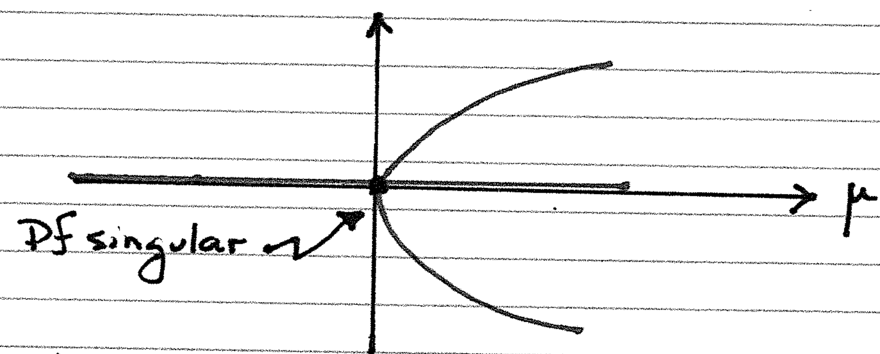
EXAMPLE (Multiple Branches)

$$\dot{x}_1 = -\mu x_1 + x_2 \qquad f(x) = \begin{pmatrix} -\mu x_1 + x_2 \\ x_2 - x_1^3 \end{pmatrix}$$
$$\dot{x}_2 = x_2 - x_1^3$$

Easy to show there are 3-branches

$$\bar{x}_+ = \begin{pmatrix} \sqrt{\mu} \\ -\mu^{3/2} \end{pmatrix} \qquad \bar{x}_- = \begin{pmatrix} -\sqrt{\mu} \\ \mu^{3/2} \end{pmatrix} \qquad \bar{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Plot x_1-components of branches.

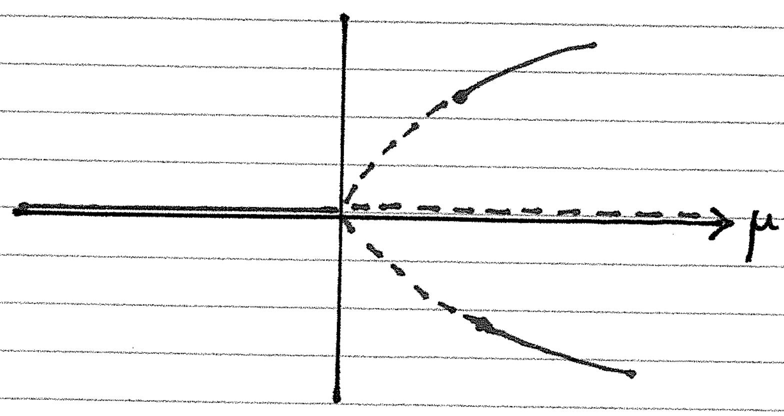


Stability not labelled

Jacobians

$$Df_{\pm} = \begin{bmatrix} -\mu & 1 \\ -3\mu & 1 \end{bmatrix} \qquad Df_0 = \begin{bmatrix} -\mu & 1 \\ 0 & 1 \end{bmatrix}$$

At the bifurcation point $\mu^* = 0$, $Df_{\pm}$ are singular. There, the fixed point is not hyperbolic. Can show



Ask me to verify this in class.

$\Uparrow$

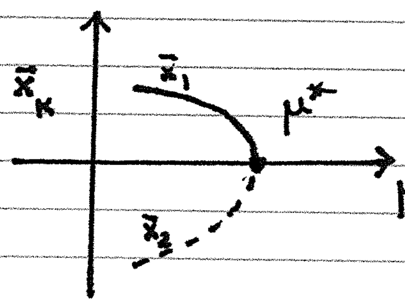
Fixed point stability

$$\dot{x} = f(x, \mu) \qquad \mu \in \mathbb{R}$$

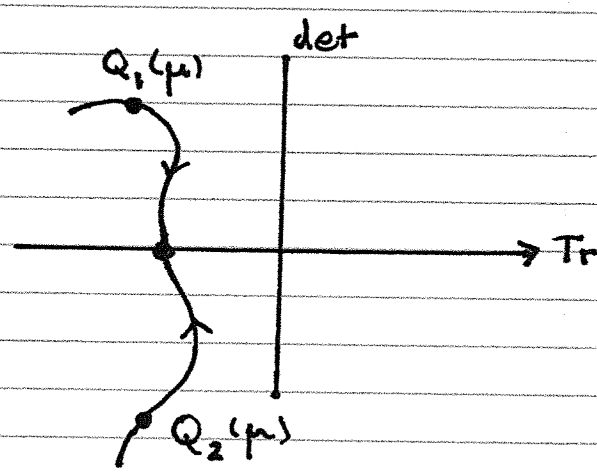
may have several branches $\bar{x}_k(\mu)$, $k = 1, 2 \ldots n$.

As one varies the parameter μ the location and stability of $\bar{x}_k$ can change. The stability can be ascertained by Jacobian $Df(\bar{x}_k)$ and trace $\operatorname{Tr} Df(\bar{x}_k)$. Toward this end define

$$Q_k(\mu) = \left(\operatorname{Tr} Df(\bar{x}_k(\mu), \mu) \; , \; \det Df(\bar{x}_k(\mu), \mu) \right)$$



location



stability

An example of a bifurcation at $\mu = \mu^*$

EXAMPLE Saddle-Node bifurcation

$$\dot{x}_1 = \mu - x_1^2 \qquad (1)$$
$$\dot{x}_2 = -x_2 \qquad (2)$$

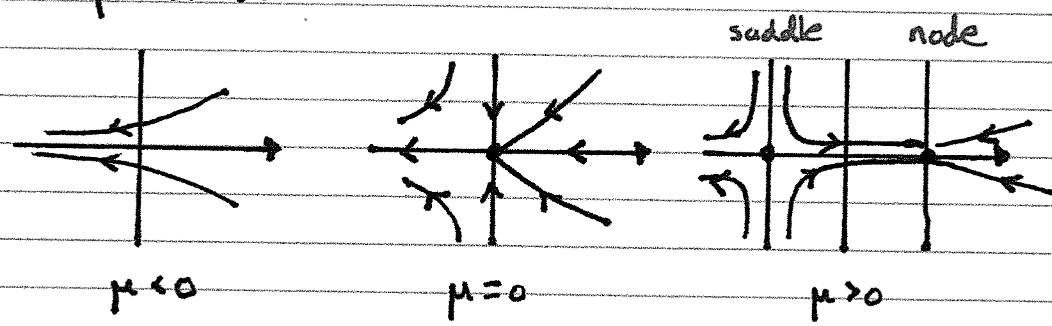
Two branches of fixed points

$$\bar{x}_+ = \begin{pmatrix} \sqrt{\mu} \\ 0 \end{pmatrix} \qquad \bar{x}_- = \begin{pmatrix} -\sqrt{\mu} \\ 0 \end{pmatrix}$$

Jacobian

$$Df(\bar{x}_\pm) = \begin{bmatrix} \mp 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix} \qquad (3)$$

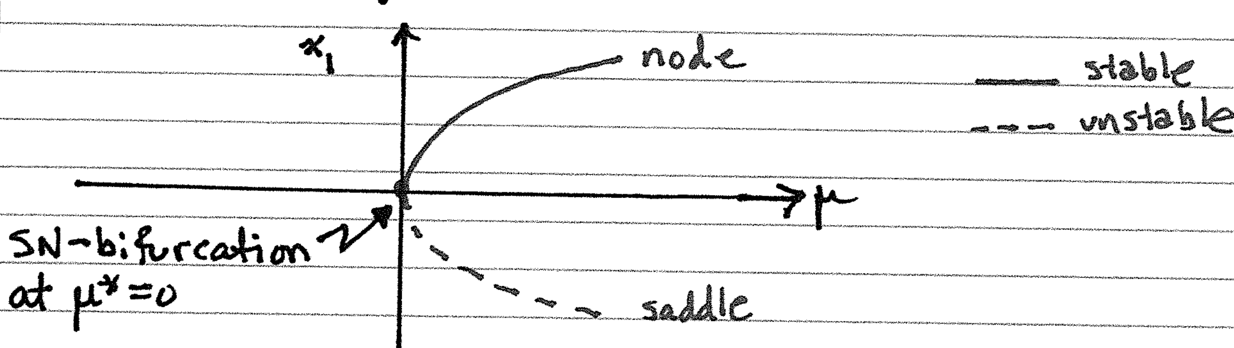
Phase portrait



From (3) we see (for $\mu > 0$)

$\bar{x}_+(\mu)$ stable node $\qquad \bar{x}_-(\mu)$ saddle

Bifurcation diagram.



EXAMPLE Saddle-node

$$(1) \qquad \dot{x}_1 = x_2 - 2x_1$$
$$(2) \qquad \dot{x}_2 = \mu + x_1^2 - x_2$$

Has two branches of fixed points where

$$x_1^2 - 2x_1 + \mu = 0$$

$$(3) \qquad x_1 = 1 \pm \sqrt{1-\mu} \qquad\qquad \mu \le 1$$

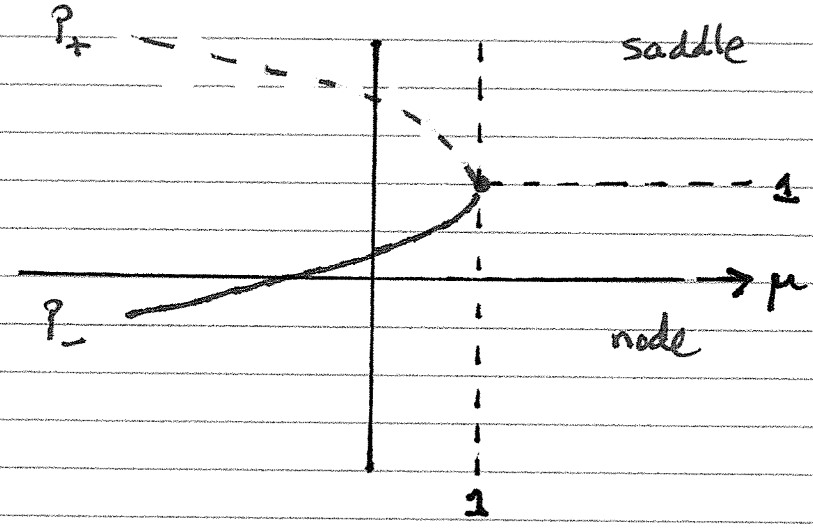
and $x_2 = 2x_1$. Let $P_\pm$ be the respective fixed pts. Since

$$\det DF(P_+) = -2\sqrt{1-\mu} < 0$$

then P_+ is a saddle $\forall \mu \le 1$. Then note

$$\det DF(P_-) = 2\sqrt{1-\mu} > 0 \qquad \mathrm{Tr}\, Df(P_1) = -3 < 0$$

so P_- is stable (node and/or spiral)



EXAMPLE (Transcritical) Saddle-Saddle

$$(1) \qquad \dot{x} = x(\mu - x)$$
$$(2) \qquad \dot{y} = -y(\mu - 2x)$$

has two fixed point branches that persist $\forall \mu \in \mathbb{R}$

$$\underline{X}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \underline{X}_1 = \begin{pmatrix} \mu \\ 0 \end{pmatrix}$$

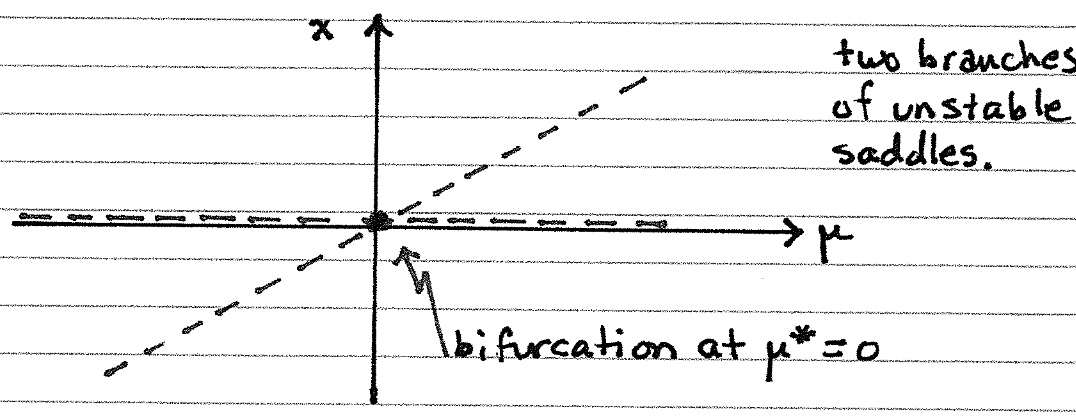
The Jacobian of (1)-(2) is

$$Df = \begin{bmatrix} (\mu - 2x) & 0 \\ 2y & -(\mu - 2x) \end{bmatrix}$$

yields

$$Df(\underline{X}_0) = \begin{bmatrix} \mu & 0 \\ 0 & -\mu \end{bmatrix} \qquad Df(\underline{X}_1) = \begin{bmatrix} -\mu & 0 \\ 0 & \mu \end{bmatrix}$$

Both have real opposite sign eigenvalues hence $\underline{X}_0$ and $\underline{X}_1$ are saddles $\forall \mu \neq 0$.



EXAMPLE Saddle-Saddle collision

$$\dot{x} = x^2 - \mu \qquad (1)$$
$$\dot{y} = -2xy \qquad (2)$$

has two branches of fixed points that exist only for $\mu \geq 0$

$$\underline{X}_+ = \begin{pmatrix} \sqrt{\mu} \\ 0 \end{pmatrix} \qquad \underline{X}_- = \begin{pmatrix} -\sqrt{\mu} \\ 0 \end{pmatrix}$$

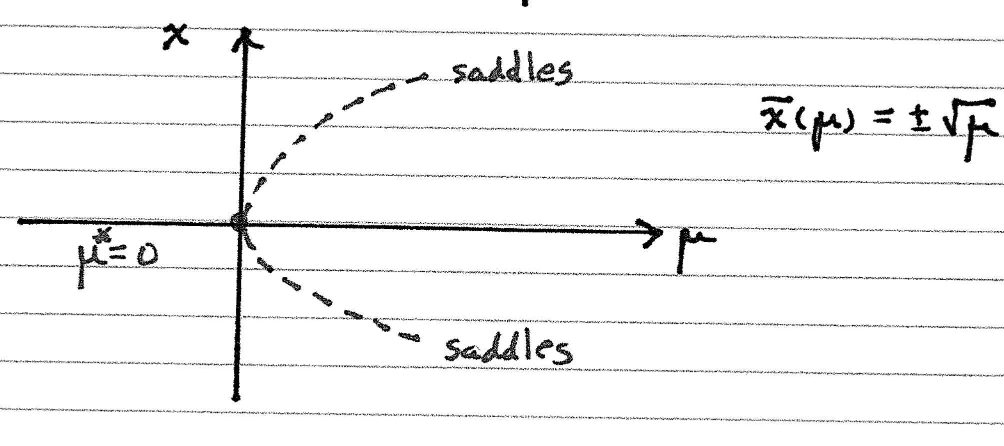
Jacobian is

$$Df = \begin{bmatrix} 2x & 0 \\ -2y & -2x \end{bmatrix}$$

It is easy to deduce

$$Df(\underline{X}_+) = \begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -2\sqrt{\mu} \end{bmatrix} \qquad Df(\underline{X}_-) = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & 2\sqrt{\mu} \end{bmatrix}$$

hence $\underline{X}_\pm$ are both saddles $\forall \mu \geq 0$.



This is <u>not</u> a Saddle-Node bifurcation