Bifurcations - overview

\[ x = f(x, \mu) \quad x \in \mathbb{R}^2 \quad \mu \in \mathbb{R}^m \]

Typically \( m = 1, 2 \) for the parameter \( \mu \).
As \( \mu \) is varied the stability and
number of the following may change.

(i) Equilibria/fixed point
(ii) Periodic orbits
(iii) Hetero and homoclinic orbits

When \( \mu \in \mathbb{R} \) it is common to summarize
such changes in a bifurcation diagram.
To make a 2D diagram one plots a
measure of \( x \) versus \( \mu \), such as

\[ x_k \text{- component, } \|x\| \text{- norm, } \max_t x_k(t) \]

For instance

\[ \begin{align*}
  \dot{x} &= \mu - x^2 \\
  \dot{y} &= 1 - y
\end{align*} \]

\[ x^*(\mu) = \pm \sqrt{\mu} \]
\[ y^*(\mu) = 1 \]

\[ \|x\| = \sqrt{x^2 + y^2} \]
Bifurcation of Equilibria of \( x = f(x, \mu) \), \( \mu \in \mathbb{R} \)

A branch \( \bar{x}(\mu) \) of fixed points is a function that satisfies

\[
(1) \quad f(\bar{x}(\mu), \mu) = 0
\]

\( \forall \mu \in I \) (some interval). There may be many different branches on the same \( I \). Differentiate (1) in \( \mu \)

\[
(2) \quad Df(\bar{x}, \mu) \frac{d\bar{x}}{d\mu} + \frac{\partial}{\partial \mu} f(\bar{x}, \mu) = 0
\]

This implies a branch persists (IFT) so long as the Jacobian \( Df(\bar{x}, \mu) \) nonsingular. In this case \( \bar{x}'(\mu) \) is well defined

\[
\frac{d\bar{x}}{d\mu} = -Df(\bar{x}, \mu)^{-1} f(\bar{x}, \mu)
\]

\[\xymatrix{
\mathbb{R} & \mathbb{R}^2 \\
\mu \ar@{^{(}->}[u] \ar[r] & x^1 \ar@{^{(}->}[u] \\
& \bar{x}_\pm(\mu) \ar@{^{(}->}[u] \\
& Df \text{ sing} \ar@{^{(}->}[u] \\
& Df \text{ not sing.} \ar@{^{(}->}[u]}
\]

shows two branches collide when \( \bar{x}'(\mu) \) not defined. \( Df \) singular.
**Example (Multiple Branches)**

\[
\begin{align*}
\dot{x}_1 &= -\mu x_1 + x_2 \\
\dot{x}_2 &= x_2 - x_1^3 \\
f(x) &= \begin{pmatrix} -\mu x_1 + x_2 \\ x_2 - x_1^3 \end{pmatrix}
\end{align*}
\]

Easy to show there are 3 branches:

\[
\vec{x}_+ = \begin{pmatrix} \sqrt{\mu} \\ -\mu^{3/2} \end{pmatrix} \quad \vec{x}_- = \begin{pmatrix} -\sqrt{\mu} \\ \mu^{3/2} \end{pmatrix} \quad \vec{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Plot \(x_1\)-components of branches.

![Graph with \(x_1\)-components of branches and stability not labelled.]

**Jacobian**

\[
\begin{align*}
Df_+ &= \begin{pmatrix} -\mu & 1 \\ -3\mu & 1 \end{pmatrix} \\
Df_0 &= \begin{pmatrix} -\mu & 1 \\ 0 & 1 \end{pmatrix}
\end{align*}
\]

At the bifurcation point \(\mu^* = 0\), \(Df_+\) are singular. There, the fixed point is not hyperbolic. Can show

![Graph with \(x_1\)-components of branches and stability not labelled.]

Ask me to verify this in class.
Fixed point stability

\[ \dot{x} = f(x, \mu) \quad \mu \in \mathbb{R} \]

may have several branches \( \tilde{x}_k(\mu) \), \( k = 1, 2, \ldots, n \).

As one varies the parameter \( \mu \), the location and stability of \( \tilde{x}_k \) can change. The stability can be ascertained by Jacobian \( DF(\tilde{x}_k) \) and trace \( Tr DF(\tilde{x}_k) \). Toward this end define

\[ Q_k(\mu) = (Tr DF(\tilde{x}_k(\mu), \mu), det DF(\tilde{x}_k(\mu), \mu)) \]

location

stability

An example of a bifurcation at \( \mu = \mu^* \)
EXAMPLE  Saddle-Node bifurcation

(1) \[ \dot{x}_1 = \mu - x_1^2 \]
(2) \[ \dot{x}_2 = -x_2 \]

Two branches of fixed points

\[ \bar{x}_+ = \left( \sqrt{\mu} \right) \quad \bar{x}_- = \left( -\sqrt{\mu} \right) \]

Jacobian

(3) \[ Df(\bar{x}_+) = \begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix} \]

Phase portrait

\[ \mu < 0 \quad \mu = 0 \quad \mu > 0 \]

From (3) we see (for \( \mu > 0 \))

\[ \bar{x}_+ (\mu) \text{ stable node} \quad \bar{x}_- (\mu) \text{ saddle} \]

Bifurcation diagram.

\( \mu^* \) saddle

SN-bifurcation at \( \mu^* = 0 \)
**Example**  
**Saddle-node**

(1) \[ x_1 = x_2 - 2x_1 \]
(2) \[ \dot{x}_2 = \mu + x_1^2 - x_2 \]

Has two branches of fixed points where

\[ x_1^2 - 2x_1 + \mu = 0 \]

(3) \[ x_1 = 1 \pm \sqrt{1-\mu} \quad \mu \leq 1 \]

and \( x_2 = 2x_1 \). Let \( P_\pm \) be the respective fixed pts. Since

\[ \det DF(P_+) = -2\sqrt{1-\mu} < 0 \]

then \( P_+ \) is a saddle \( \forall \mu \leq 1 \). Then note

\[ \det DF(P_-) = 2\sqrt{1-\mu} > 0 \quad \text{Tr} DF(P_-) = -3 < 0 \]

so \( P_- \) is stable (node and/or spiral)
**Example** (Transcritical) Saddle-Saddle

\[ \begin{align*}
(1) \quad \dot{x} &= x (\mu - x) \\
(2) \quad \dot{y} &= -y (\mu - 2x)
\end{align*} \]

has two fixed point branches that persist \( \forall \mu \in \mathbb{R} \)

\[\begin{pmatrix}
X_0 = (0) \\
X_1 = (\mu)
\end{pmatrix}\]

The Jacobian of \((1)-(2)\) is

\[Df = \begin{bmatrix}
(\mu - 2x) & 0 \\
2y & -(\mu - 2x)
\end{bmatrix}\]

yields

\[Df(X_0) = \begin{bmatrix}
\mu & 0 \\
0 & -\mu
\end{bmatrix} \quad Df(X_1) = \begin{bmatrix}
-\mu & 0 \\
0 & \mu
\end{bmatrix}\]

Both have real, opposite sign eigenvalues hence \(X_0\) and \(X_1\) are saddles \(\forall \mu \neq 0\).

\[\begin{array}{c}
\text{two branches} \\
of \text{unstable} \\
saddles,
\end{array}\]

\[\text{bifurcation at } \mu^* = 0\]
**Example**

**Saddle-Saddle collision**

(1) \[ \dot{x} = x^2 - \mu \]

(2) \[ \dot{y} = -2xy \]

has two branches of fixed points that exist only for \( \mu > 0 \)

\[ \mathbf{X}_+ = \begin{pmatrix} \sqrt{\mu} \\ 0 \end{pmatrix} \quad \mathbf{X}_- = \begin{pmatrix} -\sqrt{\mu} \\ 0 \end{pmatrix} \]

Jacobian is

\[ Df = \begin{bmatrix} 2x & 0 \\ -2y & -2x \end{bmatrix} \]

It is easy to deduce

\[ Df(\mathbf{X}_+) = \begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -2\sqrt{\mu} \end{bmatrix} \quad Df(\mathbf{X}_-) = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & 2\sqrt{\mu} \end{bmatrix} \]

hence \( \mathbf{X}_\pm \) are both saddles \( \forall \mu > 0 \).

\[ \mathbf{x} \]

saddles

\[ \mathbf{x}(\mu) = \pm \sqrt{\mu} \]

\[ \mu = 0 \]

\[ \mu \rightarrow \infty \]

saddles

This is **not** a Saddle-Node bifurcation.