# Hopf Bifurcation - intro

$$\dot{x} = f(x, \mu) \qquad x \in \mathbb{R}^2 \qquad \mu \in \mathbb{R}$$

Let $\bar{x}(\mu)$ be a branch of fixed points whose Jacobian $Df(\bar{x}, \mu)$ has complex conjugate eigenvalues

$$\lambda_{\pm}(\mu) = \alpha(\mu) \pm \beta(\mu)$$

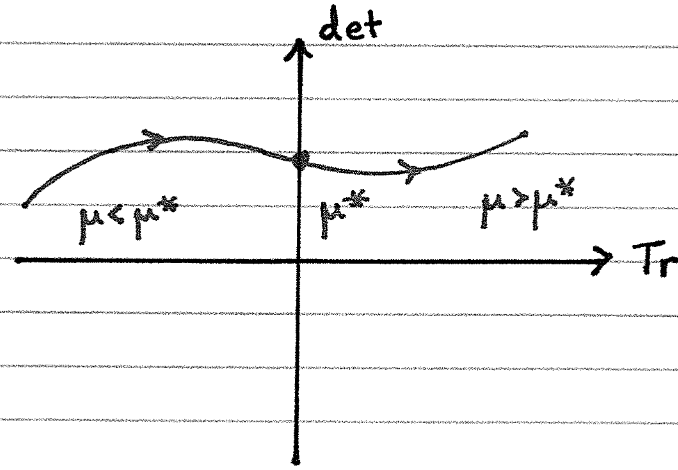
Further suppose $\exists \mu^*$ such that

$$\mathbb{R}e(\lambda_{\pm}) = \alpha(\mu^*) = 0$$

so that $\bar{x}(\mu^*)$ is a nonhyperbolic center. As $\mu$ varies the path of

$$Q_+ = (\mathrm{Tr}\, Df, \det Df)$$

may look like




Hopf bifurcation at $\mu = \mu^*$

{FIG 1}

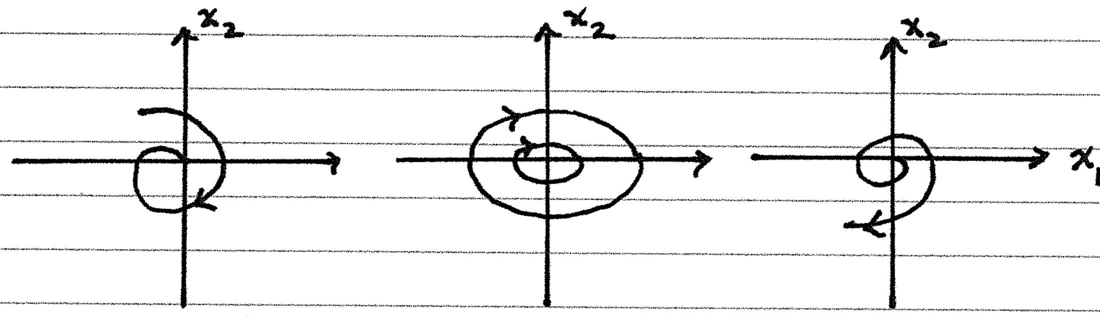
In particular recall that

$$\lambda_{\pm} = \frac{1}{2}\left(\mathrm{Tr}Df \pm \sqrt{(\mathrm{Tr}Df)^2 - 4\det Df}\ \right)$$

so the real part of $\lambda$ is

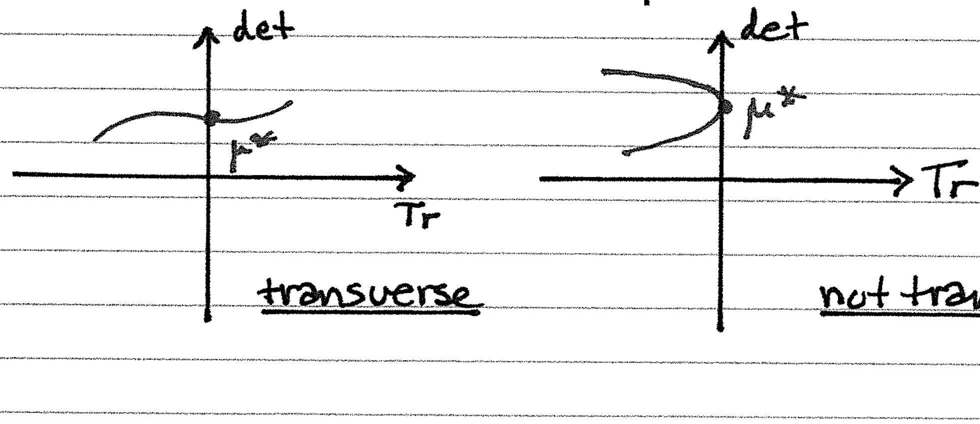
$$\alpha(\mu) = \frac{1}{2}\,\mathrm{Tr}\,Df$$

The sign of this determines the stability of the fixed points $\bar{x}(\mu)$. For the previous figure the phase portraits look like




$\mu < \mu^*$ $\qquad$ $\mu = \mu^*$ $\qquad$ $\mu > \mu^*$

So long as the crossing in FIG 1 is <u>transverse</u> a Theorem implies a family of periodic orbits emerges from the Hopf point $(\mu^*, x^*)$. We illustrate a simple example next



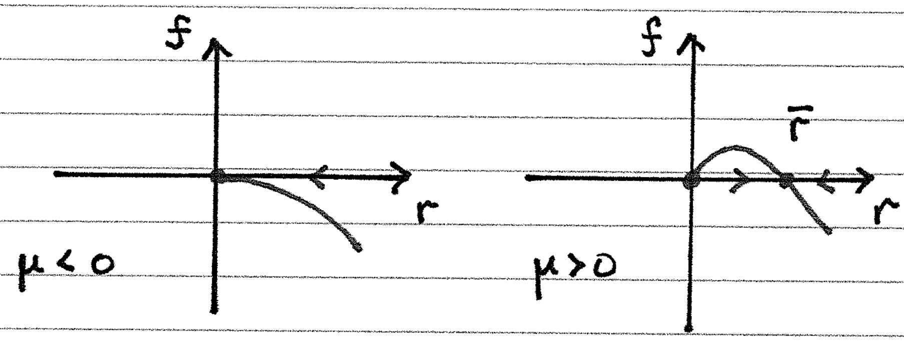

<u>transverse</u> $\qquad$ <u>not transverse</u>

## EXAMPLE   Polar

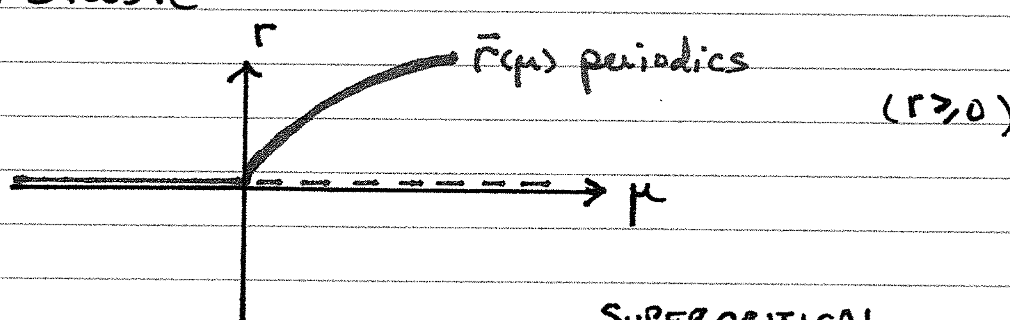
$$\dot{r} = r(\mu - r^2) \quad = f(r,\mu)$$
$$\dot{\theta} = \omega + br^2$$

Sole fixed point $\bar{x}(\mu) = 0$ for all $\mu$. A branch of periodics exists for $\mu > 0$

$$\bar{r}(\mu) = \sqrt{\mu}$$

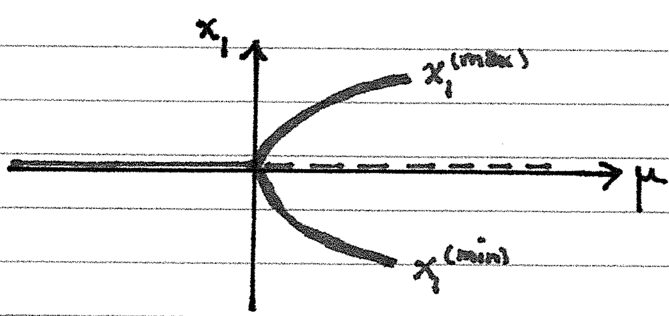



From this we deduce the limit cycles for $\mu > 0$ are all stable



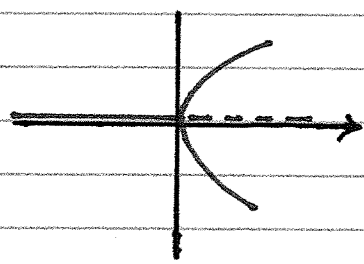

Bifurcation Diagrams

SUPERCRITICAL

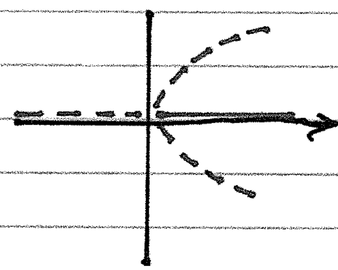


## Criticality

Has to do with whether the limit cycles emanating from the Hopf point are stable or not:



stable periods
SUPERCRITICAL



unstable periodics
SUBCRITICAL

## Theorem (Hopf)

Let $\dot{x} = f(x, \mu)$ be smooth and $f(\bar{x}(\mu), \mu) = 0$ $\forall \mu \in N_\delta(\mu^*)$ where $x^* = \bar{x}(\mu^*)$. If

(1) $\operatorname{Tr} Df(x^*, \mu^*) = 0$

(2) $\det Df(x^*, \mu^*) > 0$

(3) $\frac{d}{d\mu} \operatorname{Tr} Df(\bar{x}, \mu)\Big|_{\mu=\mu^*} \neq 0$

then $\dot{x} = f(x, \mu)$ has periodic orbits $\forall \mu \in N_\varepsilon(\mu^*)$, $\varepsilon < \delta$

**EXAMPLE** (Verifying Theorem Hypotheses)

$$\dot{x} = x^2 y + \mu y - x$$
$$\dot{y} = -x^2 y - \mu y + b$$

where $\mu, b$ are nonnegative parameters.
Seek to show the system has a Hopf bifurcation at $\mu^* = 0$. The sole fixed point is

$$\bar{x}(\mu) = \begin{pmatrix} b \\ \frac{b}{b^2+\mu} \end{pmatrix}$$

After some calculations

$$Df(\bar{x}, \mu) = \begin{bmatrix} \frac{2b^2}{b^2+\mu} - 1 & b^2+\mu \\ \frac{-2b^2}{b^2+\mu} & -(b^2+\mu) \end{bmatrix}$$

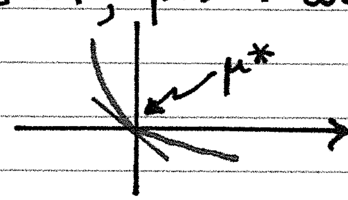
From this we then compute

(1) $$\det Df = b^2 + \mu$$

(2) $$\operatorname{Tr} Df = \frac{2b^2}{b^2+\mu} - 1 - b^2 - \mu$$

For the special case $b = 1$, $\mu > -1$ we have

$$\operatorname{Tr} Df = \frac{2}{(\mu+1)} - 2 - \mu$$




$$\frac{d}{d\mu} \operatorname{Tr} Df \Big|_{\mu^*} = -3 \neq 0$$

hence system has a Hopf bifurcation at $\mu = \mu^* = 0$

<u>EXAMPLE</u> <u>Van der Pol oscillator</u>

$$\ddot{y} - (2\mu - y^2)\dot{y} + y = 0$$

In system form

$$\begin{aligned} &(1) \qquad \dot{x}_1 = x_2 \\ &(2) \qquad \dot{x}_2 = -x_1 + 2\mu x_2 - x_1^2 x_2 \end{aligned}$$

The sole fixed point is the origin $\bar{x}(\mu) = (0,0)$
It is easy to show

$$Df(\bar{x}) = \begin{bmatrix} 0 & 1 \\ -1 & 2\mu \end{bmatrix}$$

whose eigenvalues are

$$\lambda = \mu \pm i\sqrt{1-\mu^2}$$

Clearly these are purely imaginary at $\mu^* = 0$ and the transversality condition is satisfied

$$\frac{d}{d\mu} \mathrm{Tr} Df = 2 \neq 0 \qquad \text{at } \mu = \mu^*$$

By the Hopf Theorem (1)-(2) has a Hopf bifurcation at $\mu^* = 0$ at which small amplitude periodic orbits emanate.

The theory does not predict how long (range of $\mu$) they persist.

## Hopf Bifurcations - Summary

Consider the planar system

$$\frac{dx}{dt} = f(x,y;\mu) \quad , \tag{1}$$

$$\frac{dy}{dt} = g(x,y;\mu) \quad , \tag{2}$$

where $\mu$ is a parameter. Alternately, we have the notations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}) = \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

Further, let $\bar{\mathbf{x}}(\mu) = (\bar{x}(\mu), \bar{y}(\mu))$ be the equilibria. The Jacobian of the vector field $\mathbf{f}(\mathbf{x})$ at $\bar{\mathbf{x}}$ is

$$\mathbf{Df}(\bar{\mathbf{x}}) = \begin{bmatrix} f_x(\bar{x},\bar{y}) & f_y(\bar{x},\bar{y}) \\ g_x(\bar{x},\bar{y}) & g_y(\bar{x},\bar{y}) \end{bmatrix}$$

The eigenvalues of $\mathbf{Df}(\bar{\mathbf{x}})$ are functions of the parameter $\mu$. In terms of the trace $Tr\mathbf{Df}$ and determinant $det\mathbf{Df}$, the eigenvalues of the Jacobian are:

$$\lambda_{\pm}(\mu) = \frac{Tr\mathbf{Df} \pm \sqrt{(Tr\mathbf{Df})^2 - 4det\mathbf{Df}}}{2}$$

In this summary we consider the special case where at some parameter value $\mu = \mu_0$

$$Tr\mathbf{Df}(\bar{\mathbf{x}}(\mu_0)) = 0 \tag{3}$$

$$det\mathbf{Df}(\bar{\mathbf{x}}(\mu_0)) > 0 \tag{4}$$

When these two conditions are satisfied, the eigenvalues of the Jacobian are purely imaginary. If, in addition to (3)-(4) being satisfied, the *transversality condition*

$$\frac{d}{d\mu}\{\mathbb{R}e\,(\lambda_+(\mu))\}\,|_{\mu=\mu_0} \neq 0 \tag{5}$$

is satisfied, then a *Hopf* bifurcation occurs at the bifurcation point $(\bar{\mathbf{x}}(\mu_0)), \mu_0)$ (here, $\mathbb{R}e(z)$ is the real part of $z$). At such a Hopf bifurcation for some $\mu$ near $\mu_0$, small amplitude oscillations (limit cycles) exist. The amplitude of these oscillations approaches zero as $\mu$ approaches $\mu_0$. Though Hopf theory guarantees the existence of such periodic orbits for $\mu \simeq \mu_0$, it does not guarantee the existence of the oscillations for $\mu$ further away from $\mu_0$. Often, however, the periodic orbits persist and grow in amplitude as $|\mu - \mu_0|$ increases.

At $\mu = \mu_0$ the linearized system (linearization of (1)-(2) about $\bar{\mathbf{x}}$)

$$\frac{dz}{dt} = \mathbf{Df}(\bar{\mathbf{x}})z. \quad , \quad z = (z_1, z_2) \in \mathbb{R}^2 \tag{6}$$

has a center at $z = 0$. Therefore, solutions $z(t)$ have the form

$$z(t) = c_1\vec{\zeta_1}\cos\omega t + c_2\vec{\zeta_2}\sin\omega t$$

for some real constants $c_k$ and constant vectors $\zeta_k$, $k = 1, 2$. Given the assumed conditions (3)-(4), $\lambda_\pm = \pm\omega i$ where $i^2 = -1$ and

$$\omega = \sqrt{det\mathbf{Df}} \tag{7}$$

By Hopf theory, if (3)-(5), are satisfied then for every $\mu$ with $|\mu - \mu_0|$ sufficiently small, there exists a $T$-periodic orbit (limit cycle) $\mathbf{x}_p(t;\mu)$ which satisfy (1)-(2). The period $T = T(\mu)$ and Hopf theory also guarantees

$$\lim_{|\mu-\mu_0|\to 0} T(\mu) = \frac{2\pi}{\omega} \tag{8}$$

In other words, for $\mu$ very nearly equal $\mu_0$, the period of the (emergent) periodic orbits of (1)-(2) nearly equals the period of the concentric periodic orbits of the linearized system (6).

If the Jacobian has the very special form:

$$\mathbf{Df}(\bar{\mathbf{x}}_0) = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} \quad , \quad \bar{\mathbf{x}}_0 = \bar{\mathbf{x}}(\mu_0)$$

then a <u>third-order Taylor Series</u> expansion of (1)-(2) about $\bar{\mathbf{x}}$ yields a system of the form:

$$\frac{dz_1}{dt} = (d\mu + a(z_1^2 + z_2^2))z_1 - (\omega + c\mu + b(z_1^2 + z_2^2))z_2 \tag{9}$$

$$\frac{dz_1}{dt} = (\omega + c\mu + b(z_1^2 + z_2^2))z_1 + (d\mu + a(z_1^2 + z_2^2))z_2 \tag{10}$$

which when expressed in polar coordinates is

$$\frac{dr}{dt} = (d\mu + ar^2)r \tag{11}$$

$$\frac{d\theta}{dt} = (\omega + c\mu + br^2) \tag{12}$$

for constants $a, b, c, d, \omega$, $z_1 = r\cos\theta, z_2 = r\sin\theta$. Note the equation for $r(t)$ is not coupled to the equation for $\theta$. Furthermore, depending on the signs of the constants $a$ and $d$, this third-order system possesses periodic orbits along the locus

$$\mu = -ar^2/d \quad d \neq 0$$

It can be shown that

$$d = \frac{d}{d\mu}\left\{\mathbb{R}e\left(\lambda_+(\mu)\right)\right\}|_{\mu=\mu_0}$$

so that the existence of periodic orbits local to the bifurcation point depends on $d \neq 0$. This is just the transversality condition (5).

The constant $a$ has a very complicated dependence on the vector field defining the system. In Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields, J. Guggenheimer, P. Holmes (1983) the stated value is:

$$\begin{aligned} a &= \frac{1}{16}\left[f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right] + \frac{1}{16\omega}\left[f_{xy}(f_{xx} + f_{yy})\right] \\ &- \frac{1}{16\omega}\left[g_{xy}(g_{xx} + g_{yy}) + f_{xx}g_{xx} - f_{yy}g_{yy}\right] \end{aligned}$$

evaluated at the Hopf point $(x^*, y^*, \mu^*)$. Collectively, the signs of $a$ and $d$ determine whether the Hopf bifurcation is Supercritical (stable periodics) or Subcritical (unstable periodics). Recall the locus of periodic orbit (leading-order) radii is given by

$$\mu = -ar^2/d \quad d \neq 0$$

For this reason the branch of periodic orbits are sometimes said to have a <u>quadratic tangency</u> to the fixed points.

## EXAMPLE    Multiple Hopf Points?

$$(1) \qquad \dot{x} = y$$
$$(2) \qquad \dot{y} = -F(x) - G(x,\mu)$$

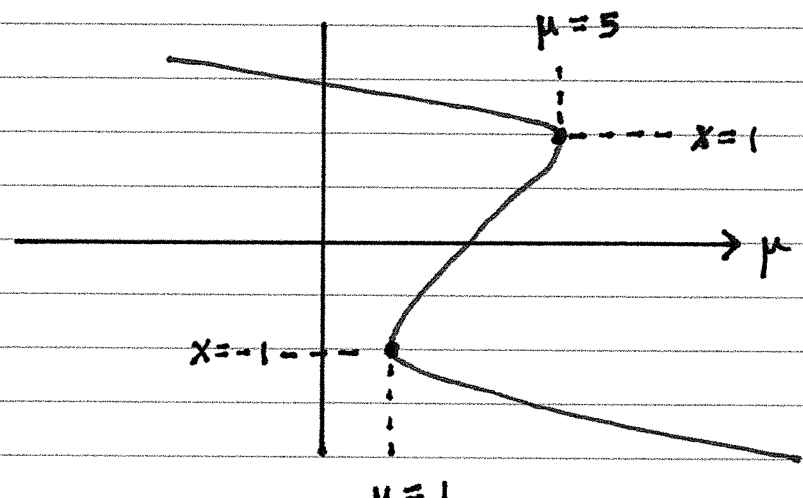
where the functions $F(x)$ and $G(x,\mu)$ are

$$F(x) = (x - \tfrac{3}{2})^2 - \tfrac{1}{4}$$
$$G(x) = \mu + x^3 - 3(x+1)$$

The fixed points of (1)-(2) are $\bar{X}(\mu) = \begin{pmatrix} \bar{x}(\mu) \\ 0 \end{pmatrix}$ where $\bar{x}$ is a root of

$$(3) \qquad \mu + \bar{x}^3 - 3(\bar{x}+1) = 0$$

The locus of these fx pts in the $(\mu, x)$-plane is:




stability not shown

Two saddle node bifurcations at

$$(\mu^*, x^*) = (1,-1), \ (5,1)$$

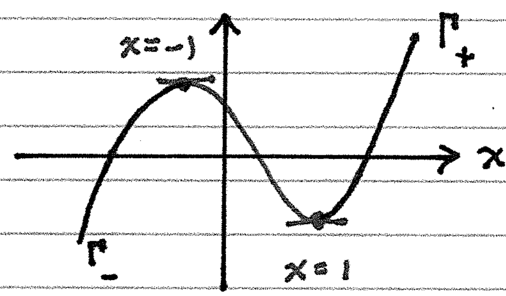
Jacobian at $\bar{X}(\mu) = \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix}$

(4) $$Df(\bar{X}) = \begin{bmatrix} 0 & 1 \\ -G'(\bar{x}) & -F(\bar{x}) \end{bmatrix}$$

(5) $$\det Df = G'(\bar{x})$$ slope of $G$ at $x$

(6) $$\mathrm{Tr}\, Df = -F(\bar{x})$$

Given the shape of $G(x) = \mu + x^3 - 3(x+1) = 0$




$G'(x) < 0$ on $x \in (-1, 1)$ means all such fixed pts are saddles.

Hopf Point Location: need three conditions

(7) $$\det Df(\bar{X}) = G'(x) > 0$$ only $\Gamma_+$

(8) $$\mathrm{Tr}\, Df(\bar{X}) = -F(x) = 0$$ imag eval

(9) $$\mu = -x^3 + 3(x+1)$$ fx pt

The transversality condition is

$$\frac{d}{d\mu} \mathrm{Tr}\, Df \bigg|_{\mu=\mu^*} = \frac{d}{d\mu}(-F(\bar{x}))\bigg|_{\mu=\mu^*} \neq 0$$

Condition (8) means $x^*$ value at Hopf Pt is a root of

$$F(x) = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4} = 0$$

Yields two candidates (use (9) to find $x$)

(10) $\qquad (\mu^*, x^*) = (1, 2)$ and $(5, 1)$

Transversality Condition

At this stage we still don't know if the points in (10) are Hopf points. Transversality

$$\frac{d}{d\mu} F(\bar{x}) \Big|_{\mu=\mu^*} = F'(\bar{x}) \underbrace{\frac{d\bar{x}}{d\mu}}_{\text{nonzero on } G=0 \text{ locus}} \Big|_{\mu=\mu^*} \neq 0$$

hence transverse if $F'(\bar{x}) \neq 0$ at $\mu = \mu^*$

$$F'(x) = 2\left(x - \frac{3}{2}\right)$$

For both $(\mu^*, x^*)$ in (10) this doesn't vanish $\Rightarrow$ Hopf Pts