Global Bifurcations of limit cycles

Most easily examined in polar systems

\begin{align*}
(1) & \quad \dot{r} = f(r, \mu) \\
(2) & \quad \dot{\theta} = g(r, \theta, \mu)
\end{align*}

where, in particular, (1) is decoupled from (2).

There are as many bifurcation types as there are for fixed points of dynamical systems on \( \mathbb{R} \).

\[ \dot{x} = f(x, \mu) \quad x \in \mathbb{R} \]

\textbf{Saddle-Node Bifurcations of Periods (SNP)}

\text{periodic \{ \text{orbits} \}} \quad \text{snp} \quad \text{origin} \quad \mu = \mu^* = 0 \quad \text{Cycles} \ f(r, \mu) = 0.

\textbf{Associated Phase Portraits in xy-plane}

\[ \mu < \mu^* \quad \mu = \mu^* \quad \mu > \mu^* \]
EXAMPLE

\[ \dot{r} = f(r, \mu) = r(\mu + r^2 - r^4) \]
\[ \dot{\theta} = g(r) = \omega + br^2 \quad ; \omega, b > 0 \]

Note \( r = 0 \) is a fixed point since \( f(0, \mu) = 0 \), \( \forall \mu \in \mathbb{R} \). Periodic orbits correspond to positive roots of \( f(r, \mu) \). Solve \( f = 0 \) for \( \mu \) to get the SNP conditions

\[(1) \quad \bar{\mu} = r^4 - r^2 = 0 \quad \text{is a fixed point (periodics)}\]
\[(2) \quad \frac{d\bar{\mu}}{dr} = 2r(2r^2 - 1) = 0 \quad \text{SNP condition}\]

After some calculations, the \((r, \mu)\) pairs that solve (1)-(2) are

\((r, \mu) = (0, 0)\)
\((r, \mu) = \left( \frac{1}{\sqrt{2}}, \frac{1}{4} \right)\)

SNP

Stability of origin fixed point is found from

\[ f'(0, \mu) = \mu \]

hence stable if \( \mu < 0 \) and unstable if \( \mu > 0 \).
Bifurcation diagram

\[ \mu^* = -\frac{1}{4} \]

In the above \( \bar{F} = 0 \) is the sole fixed point
and the stability of the periodic branch
\( \bar{r} = r^4 - r^2 \) is determined from the graphs of \( f(r, \mu) \)

\( \mu < \mu^* \) \hspace{1cm} \mu = \mu^* \hspace{1cm} \mu \in (\mu^*, \nu) \hspace{1cm} \mu > 0 \)

Of note the phase portrait for \( \mu \in (\mu^*, 0) \) we see the system has two limit cycles.
Pitchfork of Periodics. (PFP)

Recall from dynamics on $\mathbb{R}$ that

$$\dot{x} = \mu x - x^3 \quad x \in \mathbb{R}$$

has a supercritical pitchfork bifurcation at $(\mu^*, x^*) = (0, 0)$

\[ \begin{array}{c}
\mu^* = 0 \\
\end{array} \]

This can easily be adapted to create a polar system with a Pitch Fork of Periods - PFP

**EXAMPLE**

$$\begin{align*}
\dot{r} &= f(r, \mu) = r^2 \left( \mu(r-1) - (r-1)^3 \right) \\
\dot{\theta} &= 1
\end{align*}$$

Here the sole fixed point is $r = 0$ and since (after calculations)

$$f'(0, \mu) = 0 \quad \forall \mu$$

we must ascertain its stability from graphs of $f(r, \mu)$ rather than the linear analysis in (1)
Graphs of $f(r, \mu)$

$\mu < \mu^*$ \hspace{1cm} \mu = \mu^*$ \hspace{1cm} \mu \in (0, 1) \hspace{1cm} \mu > 1$

In the above $\mu^* = 0$ is the PFP bifurcation point.

Collectively we have the following bifurcation diagram for the dynamics:

$r=1$ cycle

$\mu^* = 0$ (PFP)
**Infinite Period Bifurcation (IPB)**

1. \[ \dot{r} = f(r) = r(1-r^2) \]

2. \[ \dot{\theta} = g(\theta) = \mu - \sin \theta \]

Eqn (2) has a root depending on \( |\mu| > 1, |\mu| = 1, |\mu| < 1 \)

Dots indicate \( \theta \) values for which \( \dot{\theta} = 0 \).

- \( \mu > 1 \), \( \dot{\theta} > 0 \), \( r = 1 \) periodic orbit
- \( \mu = 1 \), \( \dot{\theta} = 0 \) when \( \theta = \frac{\pi}{2} \), \( r = 1 \) homoclinic orbit
- \( \mu \in (0, 1) \), two fixed points
  - Two heteroclinic orbits
  - SN bifurcations at \( \mu = \pm 1 \)
Poincare Maps (Return Maps). Section 8.7 (8.6) of text.

A connection between the dynamics of $\dot{x} = F(x)$ and maps $x_{n+1} = f(x_n)$, $n = 1, 2, 3, \ldots$.

**Introductory example**

(1) \[ \begin{align*}
\dot{r} &= F(r) = r(1-r) \\
\dot{\theta} &= 1
\end{align*} \]

**Phase portrait**

Section $\Sigma = \{ (x_1, x_2) : x_1 = 0, x_2 \geq 0 \}$

After 1 revolution compute location

\[ \begin{align*}
\dot{r} &= F(r) \\
\dot{\theta} &= 1
\end{align*} \]

Then $r_{n+1} = r(2\pi)$,

\[ \int_{r_n}^{r_{n+1}} \frac{dr}{F(r)} = \int_0^{2\pi} dt \]

\[ \ln \left| \frac{r_{n+1}}{1-r_{n+1}} \right| \left| \frac{r_n}{r_{n+1}} \right| = 2\pi \]

(2) \[ \left| \frac{r_{n+1}}{1-r_{n+1}} \right| = \left| \frac{r_n}{1-r_n} \right| e^{2\pi} \]

Note from phase portrait that $r_n > 1 \Rightarrow r_{n+1} > 1$. 

After 1 revolution compute location
Regardless of the value of \( r_n > 0 \) we have, solving (2),

\[
(3) \quad r_{n+1} = f(r_n) = \frac{r_n e^{2\pi}}{1 + (e^{2\pi} - 1) r_n}
\]

Below is a plot of \( f(r) \):

Above diagram is an example of cobwebbing.

Orbit of map:

\[
\{ r_n, f(r_n), f(f(r_n)), \ldots \} \\
\{ r_n, r_{n+1}, r_{n+2}, \ldots \}
\]

Note that for \( \bar{r} = 1 \),

\[
f(\bar{r}) = \bar{r}
\]

So that if \( r_n = \bar{r} = 1 \) then \( r_{n+k} = \bar{r}, \forall k \geq 1, 2, \ldots \).

Here \( \bar{r} \) is called a fixed point of the map \( f \).
Remarks

(a) Map definition depends on section \( \Sigma \) and "flow" \( \dot{x} = F(x) \).

(b) For some flows the domain of the return map \( f \) is nontrivial and important.

Here, domain \( D(f) = (a, a) \).

(c) Can extend definition to higher dimensions.

Here the return map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) where domain of \( f \) consists of \( (x_2, x_3) \) coordinates of intersection point.
More formal definitions

**Defn:** The flow $\phi$ generated by $\dot{x} = F(x)$ is the function $\phi(t, y)$ such that

1. $\frac{d\phi}{dt} = F(\phi)$
2. $\phi(0, y) = y$

**Defn:** Let $\Sigma$ be a smooth surface (which does not separate $\mathbb{R}^n$) in $\mathbb{R}^n$. Define

$$D_\Sigma(P) = \{ y \in \Sigma : \exists T > 0, \phi(T, y) \in \Sigma \}$$

and $\tau(y)$ be the minimum time such that $\phi(\tau(y), y) \in D_\Sigma(P)$. We define the return map $P : \mathbb{R}^n \to \mathbb{R}^n$ by

$$P(y) = \phi(\tau(y), y)$$

where $D_\Sigma(P)$ is the domain of $P$ (w.r.t. $\Sigma$).

**Remark (Perko 193–197)**

* Poincaré–Maps existence and smoothness established (along with $P : \mathbb{R}^2 \to \mathbb{R}^2$, $P(0)$)

$$P(0) = \exp \left\{ \int_0^T \nabla \cdot F(x(t)) \, dt \right\} < 0$$

$\Rightarrow$ stable limit cycle) near T-periodic orbit $x(t)$.

* Map in "example" is a dimensionality reduction of $P$ map above.
Theoretical Formula for return maps.

Let \( \phi(t, y) \) be the flow associated with

\[
(1) \quad \dot{x} = F(x) \quad x(0) = y
\]

where \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( \Sigma \) does not separate \( \mathbb{R}^n \).

Integrating (1),

\[
x' - x = \int_0^1 F(\phi(t, x)) \, dt
\]

so that

\[
\mathcal{P}(x) = x + \int_0^1 F(\phi(t, x)) \, dt
\]

\[
\mathcal{P} : \Sigma \rightarrow \Sigma, \quad \Sigma \subset \mathbb{R}^n
\]

\[
x' = \mathcal{P}(x)
\]

and fixed points of \( \mathcal{P} \) \( \Leftrightarrow \) periodic orbits of \( \dot{x} = F(x) \).

Linearization (perhaps later)

Stability of periodic orbits of \( \dot{x} = F(x) \) \( \Leftrightarrow \) Stability of fixed pts of \( \mathcal{P}(x) \).