Global Bifurcations of limit cycles

Most easily examined in polar systems

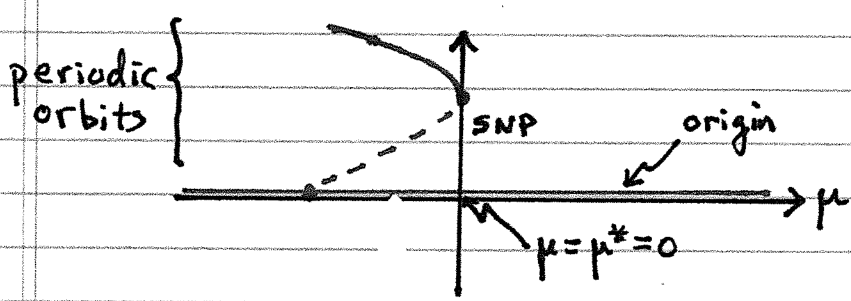
$$\dot{r} = f(r, \mu) \qquad (1)$$
$$\dot{\theta} = g(r, \theta, \mu) \qquad (2)$$

where, in particular, (1) is decoupled from (2).

There are as many bifurcation types as there are for fixed points of dynamical systems on $\mathbb{R}$.

$$\dot{x} = f(x, \mu) \qquad x \in \mathbb{R}$$

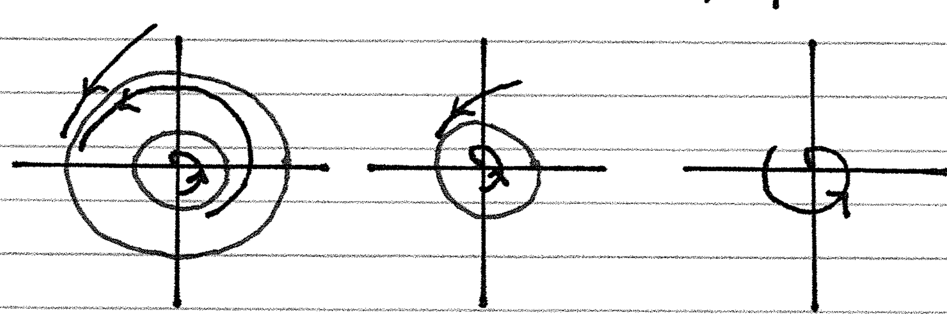
Saddle-Node Bifurcations of Periods (SNP)



$\dot{r} = f(r, \mu)$

Cycles $f(r, \mu) = 0$.

Associated Phase Portraits in xy-plane



$\mu < \mu^*$ $\qquad$ $\mu = \mu^*$ $\qquad$ $\mu > \mu^*$

EXAMPLE

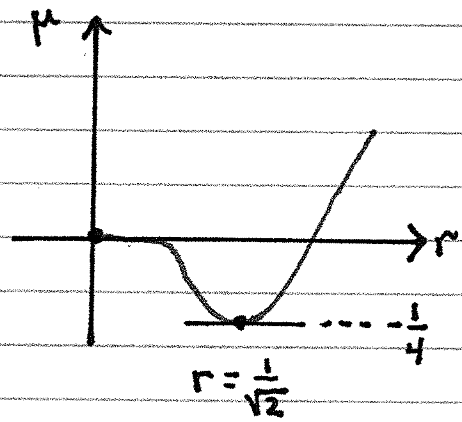
$$\boxed{\begin{aligned} \dot{r} &= f(r,\mu) = r(\mu + r^2 - r^4) \\ \dot{\theta} &= g(r) = \omega + br^2 \quad ; \omega, b > 0 \end{aligned}}$$

Note $r=0$ is a fixed point since $f(0,\mu)=0$, $\forall \mu \in \mathbb{R}$. Periodic orbits correspond to positive roots of $f(r,\mu)$. Solve $f=0$ for μ to get the SNP conditions

(1) $\quad \bar{\mu} = r^4 - r^2 = 0 \quad$ is a fixed point (periodics)

(2) $\quad \dfrac{d\bar{\mu}}{dr} = 2r(2r^2-1) = 0 \quad$ SN condition

Rotate this figure to get (μ, r) bif. diagram



After some calculations the (r,μ) pairs that solve (1)-(2) are

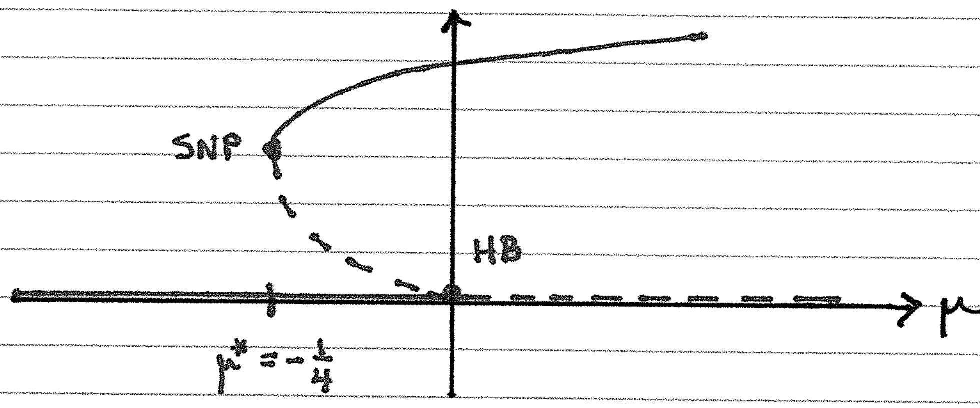
$$\underbrace{(r,\mu) = (0,0) \qquad (r,\mu) = \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{4}\right)}_{\text{SNP}}$$

Stability of origin fixed point is found from

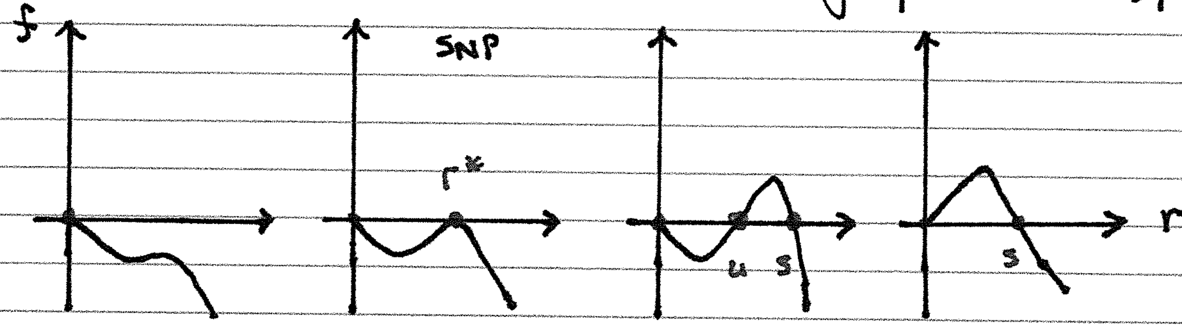
$$f'(0,\mu) = \mu$$

hence stable if <0 and unstable if >0.

Bifurcation diagram

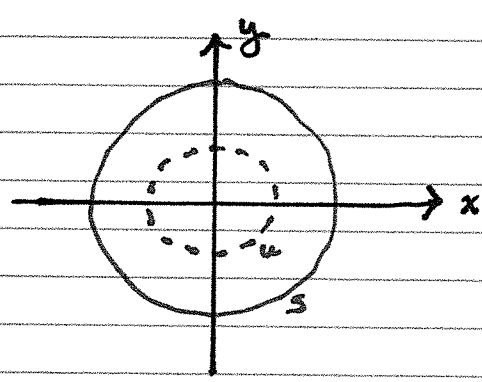


In the above $\bar{r} = 0$ is the sole fixed point and the stability of the periodic branch $\bar{\mu} = r^4 - r^2$ is determined from the graphs of $f(r,\mu)$



$\mu < \mu^*$ $\quad\quad$ $\mu = \mu^*$ $\quad\quad$ $\mu \in (\mu^*, 0)$ $\quad\quad$ $\mu > 0$

Of note the phase portrait for $\mu \in (\mu^*, 0)$ we see the system has two limit cycles.

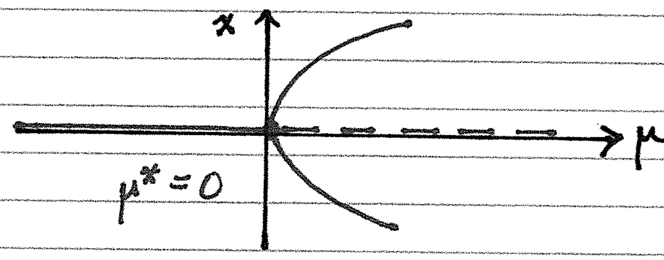


Pitchfork of Periodics (PFP)

Recall from dynamics on $\mathbb{R}$ that

$$\dot{x} = \mu x - x^3 \qquad x \in \mathbb{R}$$

has a supercritical pitchfork bifurcation at $(\mu^*, x^*) = (0, 0)$



This can easily be adapted to create a polar system with a Pitch Fork of Periods - PFP

EXAMPLE

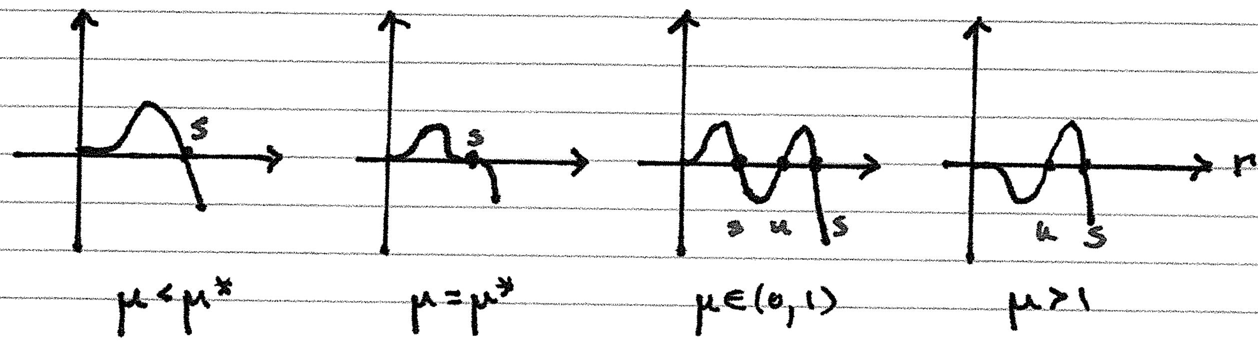
$$\begin{aligned} \dot{r} &= f(r, \mu) = r^2(\mu(r-1) - (r-1)^3) \\ \dot{\theta} &= 1 \end{aligned}$$

Here the sole fixed point is $r = 0$ and since (after calculations)

$$(1) \qquad f'(0, \mu) = 0 \qquad \forall \mu$$

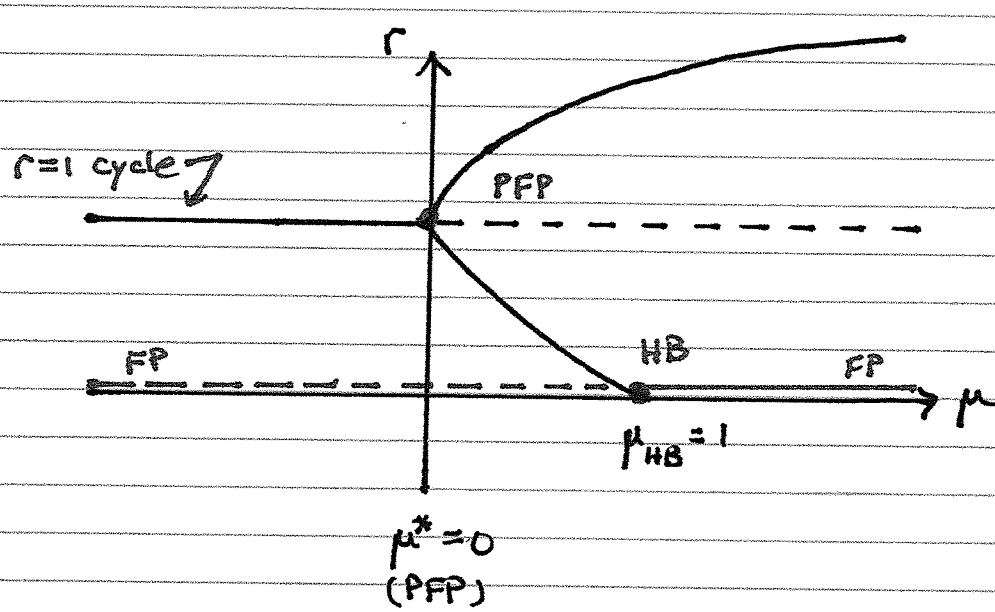
we must ascertain its stability from graphs of $f(r, \mu)$ rather than the linear analysis in (1)

Graphs of $f(r,\mu)$



In the above $\mu^*=0$ is the PFP bifurcation point

Collectively we have the following bifurcation diagram for the dynamics

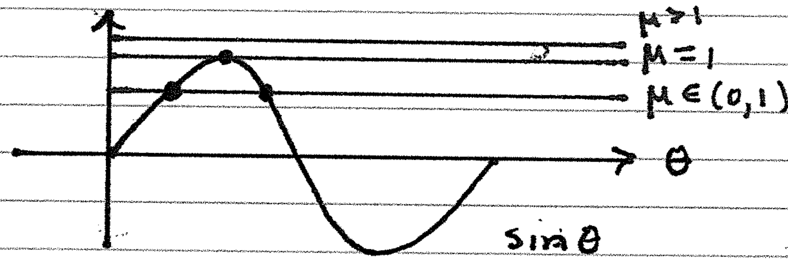


Infinite Period Bifurcation (IPB)

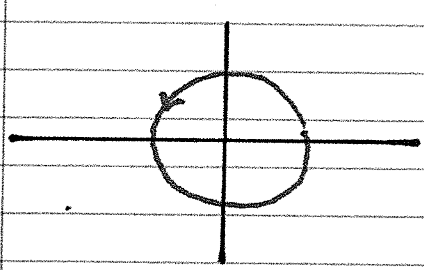
$$(1) \qquad \dot{r} = f(r) = r(1-r^2)$$

$$(2) \qquad \dot{\theta} = g(\theta) = \mu - \sin\theta$$

Eqn (2) has a root depending on $|\mu|>1$, $|\mu|=1$, $|\mu|<1$

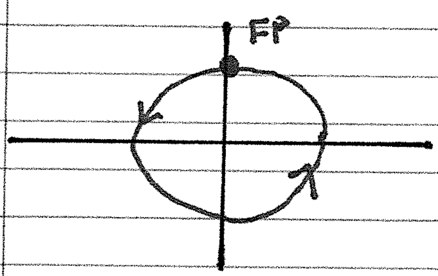


Dots indicate θ values for which $\dot{\theta}=0$.



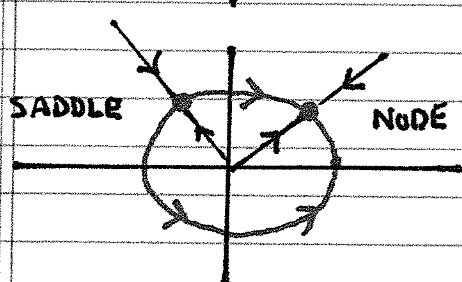
$\mu > 1$

$\dot{\theta} > 0$
$r=1$ periodic orbit



$\mu = 1$

$\dot{\theta} = 0$ when $\theta = \pi/2$
$r=1$ homoclinic orbit



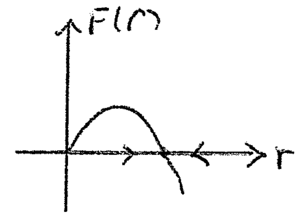
$\mu \in (0,1)$

Two fixed points
Two heteroclinic orbits
SN bifurcations at $\mu = \pm 1$

Poincare Maps (Return Maps) Section 8.7 (8.6) of text.

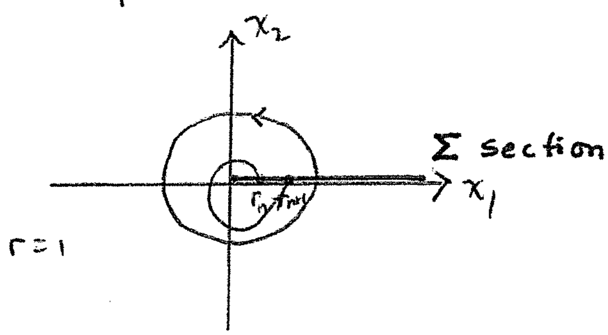
A connection between the dynamics of $\dot{x} = F(x)$ and maps $x_{n+1} = f(x_n)$, $n = 1, 2, 3 \ldots$

Introductory example



$$(1) \qquad \begin{aligned} \dot{r} &= F(r) = r(1-r) \\ \dot{\theta} &= 1 \end{aligned}$$

Phase portrait



Section $\Sigma = \{(x_1, x_2) : x_2 = 0,\ x_1 \geq 0\}$

After 1 revolution compute location

$$\begin{aligned} \dot{r} &= F(r) \qquad & r(0) &= r_n \\ \dot{\theta} &= 1 & \theta(0) &= 0 \end{aligned}$$

Then $r_{n+1} = r(2\pi)$.

$$\int_{r_n}^{r_{n+1}} \frac{dr}{F(r)} = \int_0^{2\pi} dt$$

$$\ln\left|\frac{r}{1-r}\right| \Bigg|_{r_n}^{r_{n+1}} = 2\pi$$

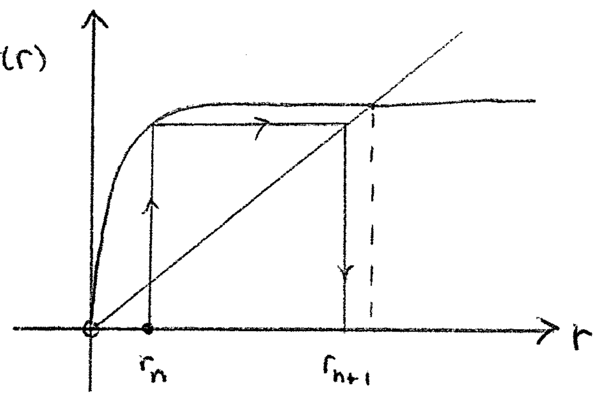
$$(2) \qquad \left|\frac{r_{n+1}}{1-r_{n+1}}\right| = \left|\frac{r_n}{1-r_n}\right| e^{2\pi}$$

Note from phase portrait that $r_n > 1 \Rightarrow r_{n+1} > 1$.

Regardless of the value of $r_n > 0$ we have, solving (2),

$$\boxed{(3) \qquad r_{n+1} = f(r_n) \equiv \frac{r_n e^{2\pi}}{1 + (e^{2\pi} - 1) r_n}}$$

Below is a plot of $f(r)$:



Poincare Map (Return Map) with respect to section Σ.

Above diagram is an example of cobwebbing.

Orbit of map:

$$\{r_n, f(r_n), f(f(r_n)), \ldots \}$$

$$\{r_n, r_{n+1}, r_{n+2}, \ldots \}$$

Note that for $\bar{r} = 1$,

$$f(\bar{r}) = \bar{r}$$

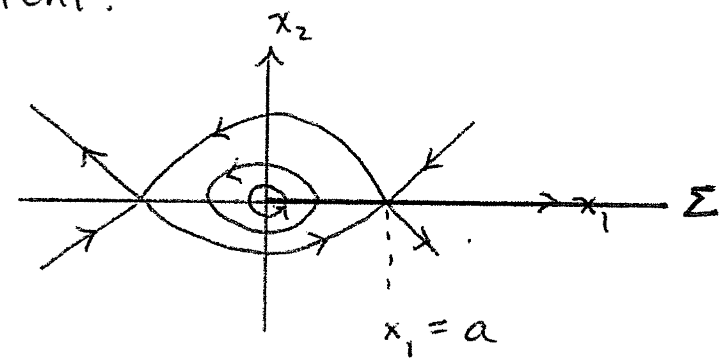
So that if $r_n = \bar{r} = 1$ then $r_{n+K} = \bar{r}$, $\forall K = 1, 2 \ldots$.

Here $\bar{r}$ is called a fixed point of the map f.

Remarks

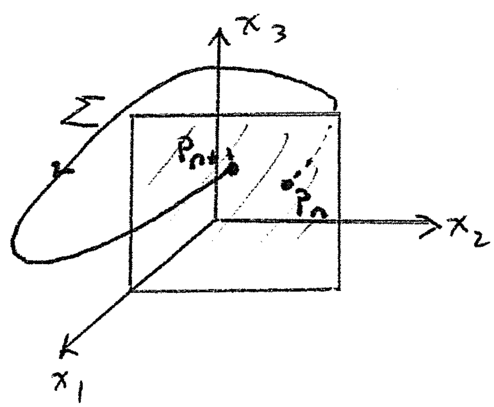
(a) Map definition depends on section Σ and "flow" $\dot{x} = F(x)$.

(b) For some flows the domain of the return map f is nontrivial and important!



here, domain $D(f) = (0, a)$.

(c) Can extend definition to higher dimensions.



$\dot{x} = F(x)$

$x(t) \in \mathbb{R}^3$

Here the return map $f: \mathbb{R}^2 \to \mathbb{R}^2$ where domain of f consists of (x_2, x_3) coordinates of intersection pt.

More formal definitions

Defn: The flow ϕ generated by $\dot{x} = F(x)$ is the function $\phi(t, y)$ such that

(1) $$\frac{d\phi}{dt} = F(\phi)$$

(2) $$\phi(0, y) = y$$

Defn: Let Σ be a smooth surface (which does not separate $\mathbb{R}^n$) in $\mathbb{R}^n$. Define

$$D_\Sigma(P) = \{ y \in \Sigma : \exists T > 0 ,\ \phi(T, y) \in \Sigma \}$$

and $\tau(y)$ be the minimum time such that $\phi(\tau(y), y) \in \Sigma$, $y \in D_\Sigma(P)$. The we define the return map $P : \mathbb{R}^n \to \mathbb{R}^n$ by

$$P(y) = \phi(\tau(y), y)$$

where $D_\Sigma(P)$ is the domain of P (w.r.t. Σ).

Remark (Perko 193 – 197)

* Poincaré-Maps existence and smoothness established (along with $P : \mathbb{R}^2 \to \mathbb{R}^2$, $P'(0)$.

$$P'(0) = \exp\left\{ \int_0^T \vec{\nabla} \cdot \vec{f}(\gamma(t))\,dt \right. < 0$$

$\Rightarrow$ stable limit cycle) near T-periodic orbit $\gamma(t)$.

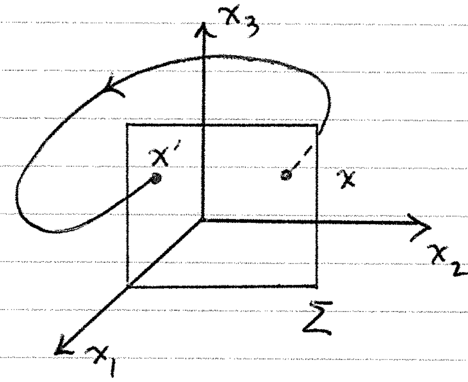
* Map in "example" is a dimensionality reduction of P map above.

Theoretical Formula for return maps.

Let $\phi(t, y)$ be the flow associated with

(1) $\quad \dot{x} = F(x) \qquad x(0) = y$

where $F: \mathbb{R}^n \to \mathbb{R}^n$ and Σ does not separate $\mathbb{R}^n$.



$x = (x_1, \ldots x_n)$

$x' = (x_1', \ldots x_n')$

Integrating (1)

$$x' - x = \int_0^{\tau(x)} F(\phi(t, x))\, dt$$

so that

$$P(x) = x + \int_0^{\tau(x)} F(\phi(t, x))\, dt$$

$$P : \Sigma \to \Sigma \quad , \quad \Sigma \subset \mathbb{R}^n$$

$$x' = P(x)$$

and fixed points of P $\Leftrightarrow$ periodic orbits of $\dot{x} = F(x)$

Linearization (perhaps later)

Stability of periodic orbits of $\dot{x} = F(x)$ $\Leftrightarrow$ Stability of fixed pts of $P(x)$.