Saddle node bifurcations

Maps can also have saddle-node, transcritical bifurcations and pitchforks. We will only discuss one

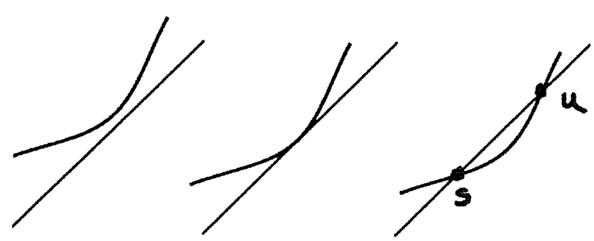
Defn: A map $x \mapsto f(x,\lambda)$ has a saddle node bifurcation at (x^*, λ^*) if $\exists \varepsilon > 0$ and open interval I s.t.

(i) $\lambda \in (\lambda^* - \varepsilon, \lambda^*) \Rightarrow f$ has no fixed pt

(ii) $\lambda = \lambda^* \Rightarrow f$ has a unique neutrally stable fixed point x^*

(iii) $\lambda \in (\lambda^*, \lambda^* + \varepsilon) \Rightarrow f$ has two fixed points, one unstable and the other (asymp) stable.

This complicated definition is simply illustrated in the figure below where $\lambda_0 = \lambda^*$.



(a) $\lambda < \lambda_0$ (b) $\lambda = \lambda_0$ (c) $\lambda > \lambda_0$

Fig. 6.4 A typical saddle-node bifurcation.

EXAMPLE: **Exponential Map**

$$f(x,\lambda) = e^x + \lambda$$

Claim f has a SN-bifurcation at $(x,\lambda) = (0,-1)$.
First define

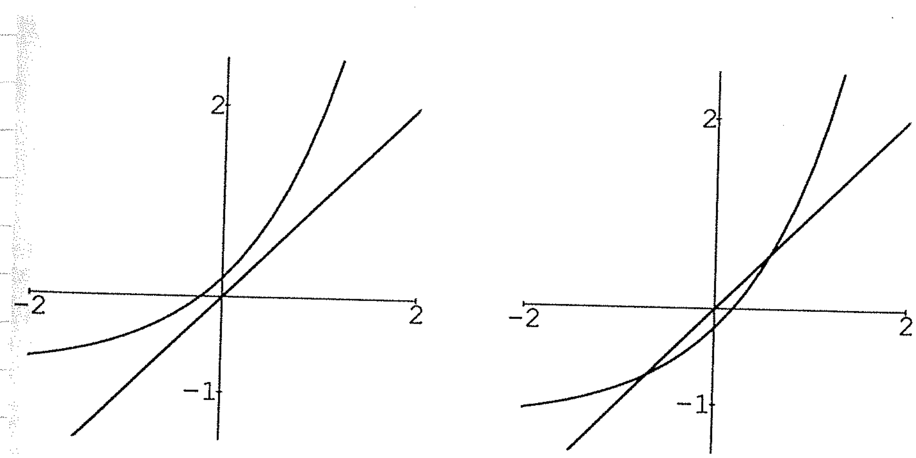
$$F = e^x - x + \lambda$$

vanishes at fixed points $\bar{x}$ of $f(x)$. Since
$F'(x) = e^x - 1 = 0 \iff x = 0$ the min of F is

$$F_{min} = F(0) = \lambda + 1$$

Thus point of tangency of $f(x)$ and x is at $\lambda = -1$.
Furthermore, $F''(x) > 0$ (concave up) for all x, λ
so $\exists$ two roots of F when $\lambda < -1$.

These observations confirm the SN:



(a) $\lambda > -1$ (b) $\lambda < -1$

Fig. 6.6 A saddle-node bifurcation in the exponential family $E_\lambda(x) = e^x + \lambda$.