Logistic Map (Single fixed point)

$$x_{n+1} = f(x_n) \qquad f(x) = \mu x(1-x)$$

where μ is a parameter. Fixed points $\bar{x}$:

$$\bar{x} = \mu \bar{x}(1-\bar{x})$$

has two solutions

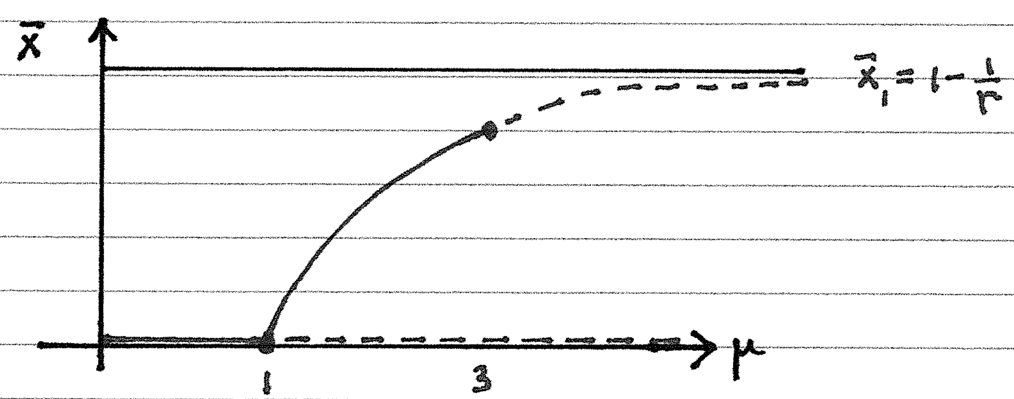
$$\bar{x}_0 = 0 \qquad \bar{x}_1 = 1 - \frac{1}{\mu}$$

Determine linear stability of these using
$f'(x) = \mu - 2\mu x$

$$f'(\bar{x}_0) = \mu \qquad f'(\bar{x}_1) = 2 - \mu$$

Summarize stability of fixed points

	(0, 1)	(1, 3)	(3, ∞)
$\bar{x}_0$	s	u	u
$\bar{x}_1$	u	s	u



Periodic Points and Orbits

Defn $x \mapsto f(x)$ has a periodic point $\bar{x}$ of minimal period k if

$$\bar{x} = f^k(\bar{x}) \tag{1}$$

and k is the smallest integer such that (1) is true.

Associated with $\bar{x}$ is the periodic orbit $\gamma(\bar{x})$ of f:

$$\gamma(\bar{x}) = \{\bar{x}, \underbrace{f(\bar{x}), f^2(\bar{x}), \ldots \underset{\overset{\shortparallel}{\bar{x}}}{f^k(\bar{x})}}_{\text{repeats after } k \text{ iterates}}, f(\bar{x}), \ldots\}$$

EXAMPLE $\gamma(\bar{x}) = \{\bar{x}, \bar{x}, \ldots\}$ is a periodic orbit of minimal period 1 if $\bar{x}$ is a fixed point of $f(x)$

EXAMPLE If

$$\gamma(a) = \{a, b, a, b, a, b \ldots\}$$

is minimal period 2 then so is

$$\gamma(b) = \{b, a, b, a, \ldots\}$$

Logistic Map - Period two orbits

$$f(x) = \mu x(1-x)$$

The second iterate map is $x \mapsto f^2(x)$

$$f^2(x) = f(f(x))$$

$$(1) \qquad f^2(x) = \mu f(x)(1 - f(x))$$

From (1) we note that if $\bar{x}$ is a fixed point of $f(x)$ it is also a fixed point of the second iterate map:

$$(2) \qquad f^2(\bar{x}) = \mu f(\bar{x})(1-f(\bar{x})) = \bar{x}$$

Such fixed points have mimimal periods of 1 and do not correspond to period 2 orbits. Explicitly

$$\boxed{f^2(x) = \mu x(1-x)(1-\mu x(1-x))}$$

we seek roots of the quantic

$$f^2(x) = x$$

given we know two roots from (2) are

$$\bar{x}_0 = 0 \qquad \bar{x}_1 = 1 - \frac{1}{r}$$

Using long division and knowing roots $\bar{x}_0$ and $\bar{x}_1$ of $f^2(x) - x$ we may write

$$f^2(x) - x = r x \left(x - 1 + \frac{1}{r}\right) Q(x)$$

where the quadratic $Q(x)$ is

$$Q(x) = -r^2 x^2 + (r+1) r x - (r+1)$$

The roots of $Q(x)$ are for $\underline{r > 3}$

$$a = \frac{1}{2r}\left\{ (r+1) + \sqrt{(r+1)(r-3)} \right\}$$

$$b = \frac{1}{2r}\left\{ (r+1) - \sqrt{(r+1)(r-3)} \right\}$$

<u>Four orbits</u>

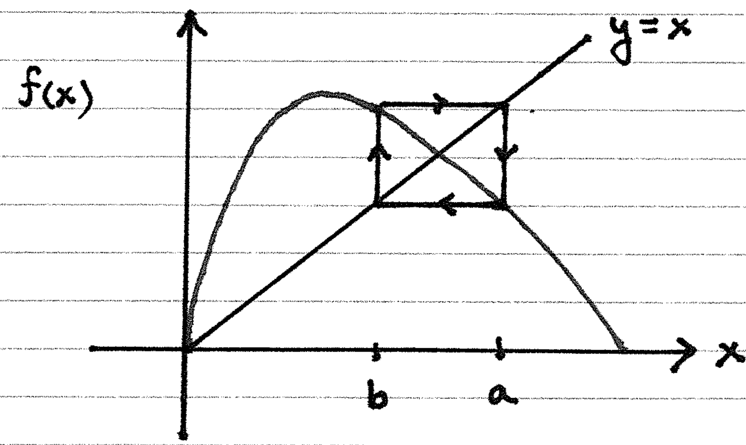
$$\gamma(\bar{x}_0) = \{0, 0, 0, 0, 0, \ldots\}$$

$$\gamma(\bar{x}_1) = \{\bar{x}_1, \bar{x}_1, \bar{x}_1, \bar{x}_1, \bar{x}_1, \ldots\}$$

$$\gamma(a) = \{a, b, a, b, \ldots\} \quad \text{period 2}$$

$$\gamma(b) = \{b, a, b, a, \ldots\} \quad \text{period 2}$$

Period 2 orbit cobweb



$a > b$

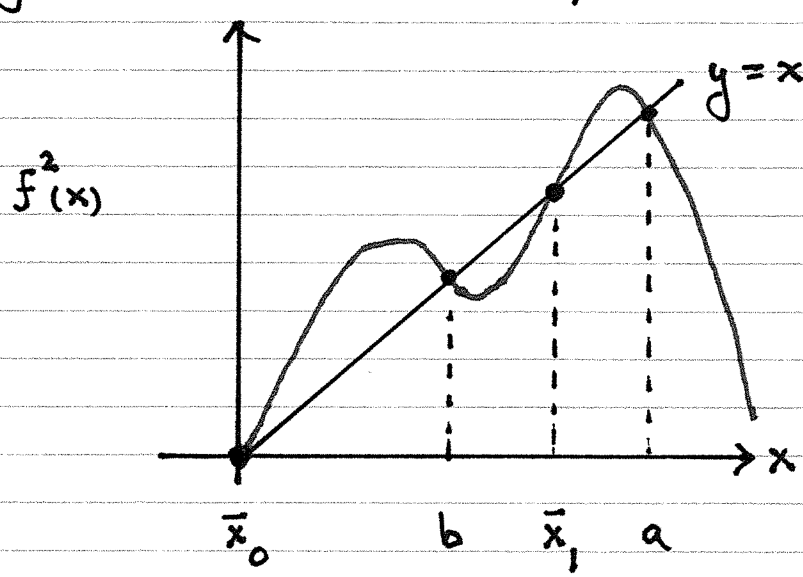
Shows a period 2 orbit

Second iterate map $x \longmapsto f^2(x)$

Given x_n this map determines x_{n+2} for all n.

$$x_{n+2} = f^2(x_n)$$

The second iterate map also has fixed points. Some are fixed points of $f(x)$ and others correspond to the values of a, b in the figure above. Qualitatively the quartic $f^2(x)$ is



Defn: Let $\gamma = \{a_1 a_2 \ldots a_k\, a_1 a_2 \ldots\}$ be a periodic orbit with minimal period k. We say γ is stable (asymptotically stable) if a_j is a stable (asymptotically stable) fixed point of $x \mapsto f^k(x)$.

Remark: This means our previous period two orbit is stable only if a, b are stable fixed points of the second iterate map

Logistic Map: Stability of period 2 orbits

Consider the stability of the fixed points a, b of second iterate map.

$$\lambda = \frac{d}{dx} f(f(x)) \Big|_{x=a,b} \qquad \text{multiplier}$$

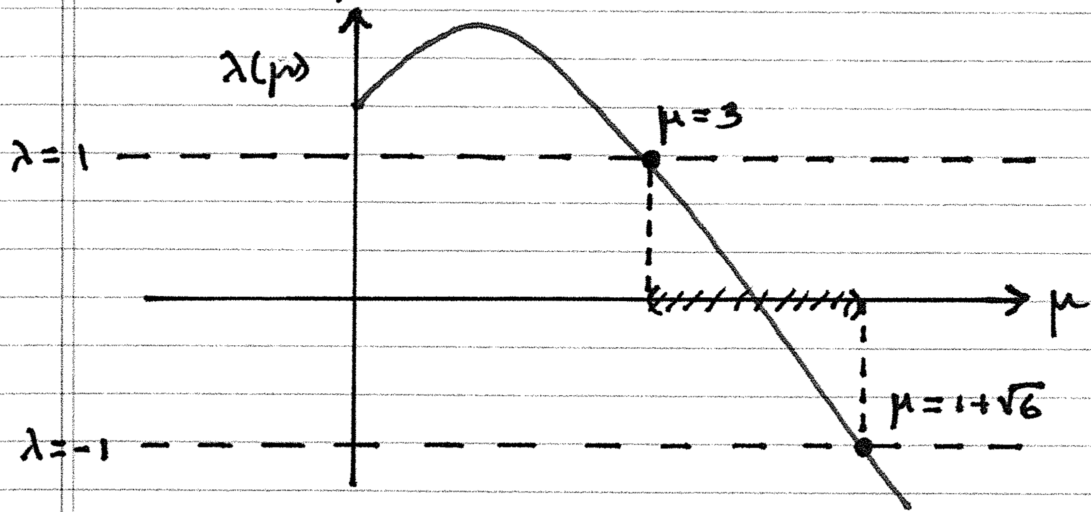
$$\lambda = f'(f(a))\, f'(a)$$

$$\boxed{\lambda = f'(a)\, f'(b)}$$

Need $|\lambda| < 1$ for asymptotic stability. Calculations

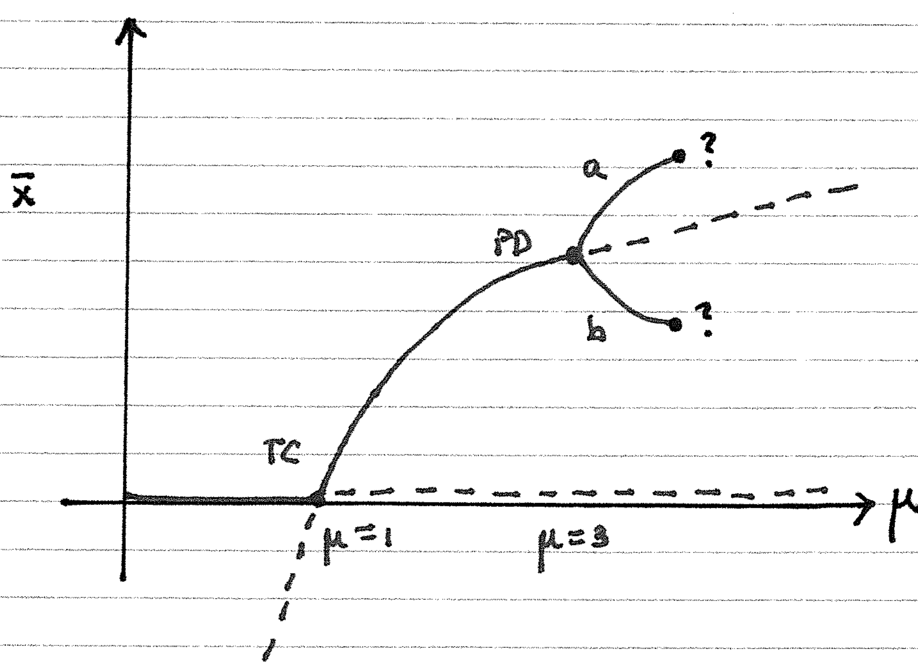
$$\boxed{\begin{aligned} \lambda &= r^2(1-2a)(1-2b) \\ \lambda &= 4 + 2r - r^2 \end{aligned}}$$

Requirement that $|\lambda| < 1$ can be determined graphically:



For $\mu \in (3, 1+\sqrt{6})$ the sole periodic (period 2) orbit exists and is stable. For $\mu > 1+\sqrt{6}$ the period 2 orbit is unstable.

Bifurcation diagram for $f(x) = \mu x(1-x)$



TC = transcritical   PD = period doubling