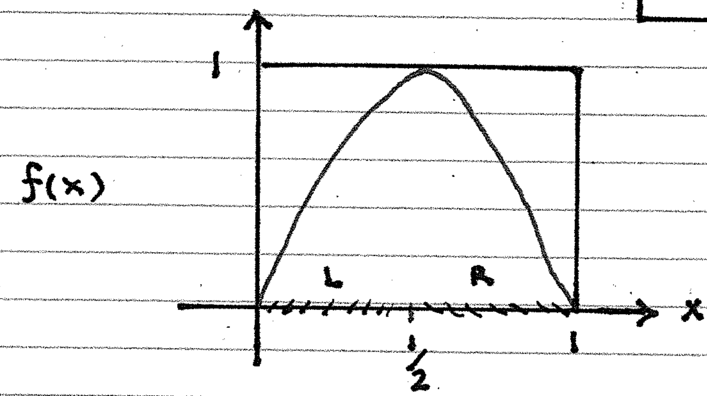
Logistic Map at $\mu = 4$

$$\boxed{f(x) = 4x(1-x)}$$



accurate graph

The maxima occurs when $f'(x) = 4 - 8x = 0$ or at $x = \frac{1}{2}$ at which $f(\frac{1}{2}) = 1$.

Some remarks:

(1) $f([0,1]) = [0,1]$ is onto

(2) Fixed points of the map $f(\bar{x}) = \bar{x}$ implies

$$\bar{x}_1 = 0 \qquad \bar{x}_2 = \frac{3}{4}$$

(3) Some special trajectories

$$\gamma(0) = \{0, 0, 0, \ldots\}$$

$$\gamma(1) = \{1, 0, 0, \ldots\}$$

Lastly for future discussion we define the intervals

$$L = [0, \tfrac{1}{2})$$

$$R = [\tfrac{1}{2}, 1]$$

Deducing the shape of $f^2(x)$

(1) $f^2(x)$ is a quartic polynomial

(2) $f^2([0,1]) = [0,1]$ since $f([0,1]) = [0,1]$

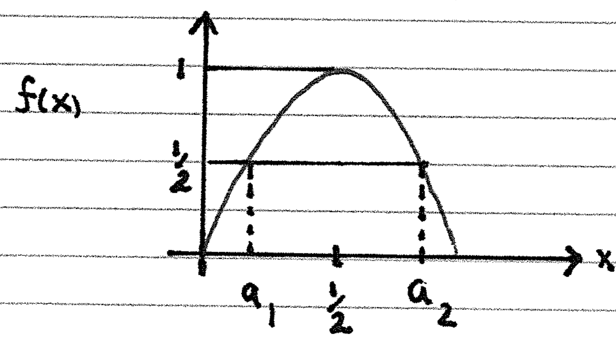
(3) $f^2(0) = f^2(\tfrac{1}{2}) = f^2(1) = 0$

(4) Fixed points $\bar{x}_1, \bar{x}_2$ of $f(x)$ are also fixed points of $f^2(x)$

Now we deduce how many $x \in [0,1]$ have

$$f^2(x) = 1$$

Can be deduced from graph of $f(x)$

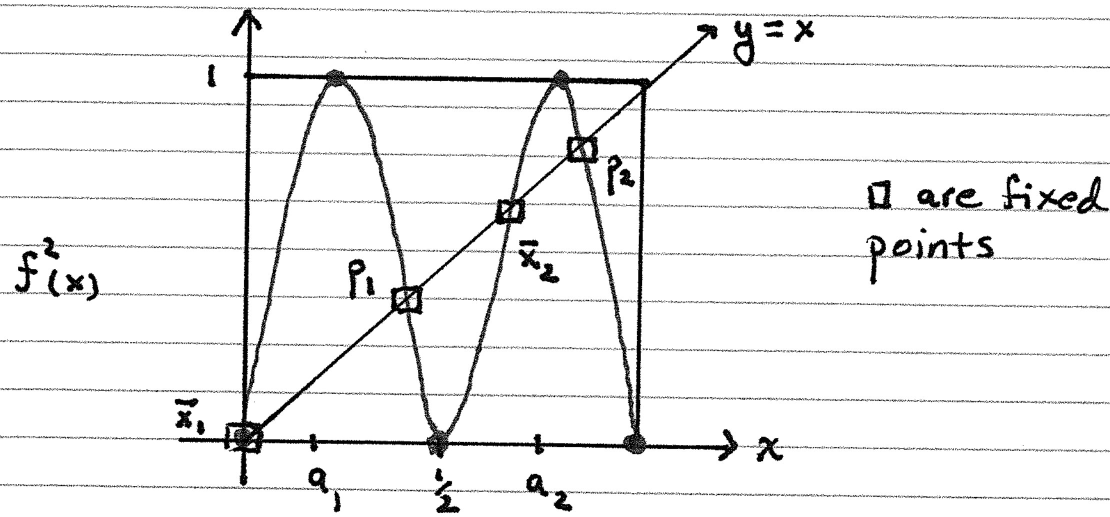


$f(a_1) = f(a_2) = \tfrac{1}{2}$

$a_1 \in L \qquad a_2 \in R$

For these two a_k we thus have

(5) $$f^2(a_1) = f^2(a_2) = 1$$



Two of these fixed points are $\bar{x}_1 = 0$, $\bar{x}_2 = 3/4$.

The other two $p_1 < p_2$ define a period 2 orbit of $f(x)$

$$\gamma(p_1) = \{ p_1, p_2, p_1, p_2, \dots \}$$

One can prove**

$$a_1 < \frac{1}{4} \qquad\qquad a_2 > \frac{3}{4} = \bar{x}_2$$

from which we deduce the order on the graph.

** Recall a_k are roots of $f(x) = \frac{1}{2}$ yields

$$x = \frac{1}{2} \pm \frac{\sqrt{8}}{8}$$

where, explicitly,

$$a_1 = \frac{1}{2} - \frac{\sqrt{8}}{8} \simeq 0.146 \qquad a_2 = \frac{1}{2} + \frac{\sqrt{8}}{8} \simeq 0.853$$

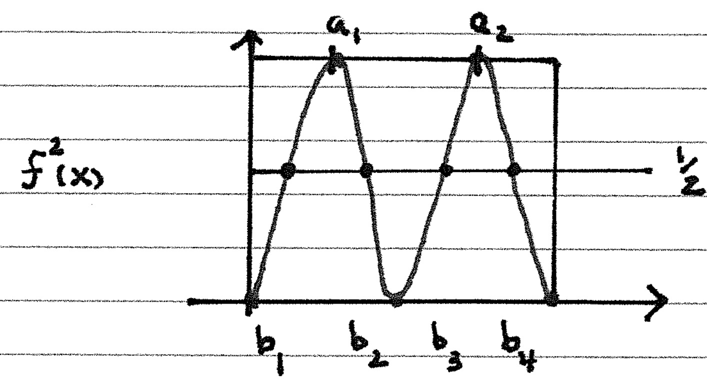
Logistic Map: Period 3 orbits

(i) $f^3(x)$ is an 8^{th} degree polynomial 2^3

(ii) $f^3([0,1]) = [0,1]$

(iii) $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{3}{4}$ are fixed pts of $f^3(x)$

Finalize deducing graph of $f^3(x)$ by finding pre images of $f^2(x) = \frac{1}{2}$ since $f(\frac{1}{2}) = 1$ tells us where $f^3(x) = 1$.

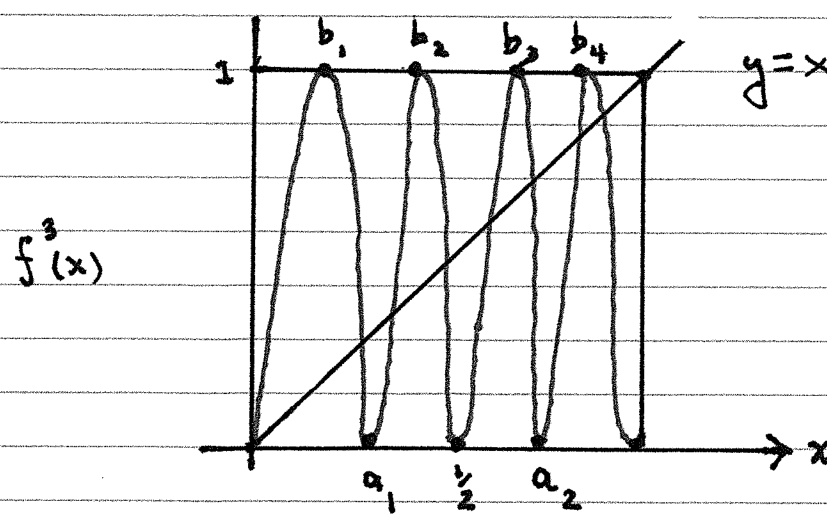


There exist $b_k \in (0,1)$ such that

$b_1 < b_2 < b_3 < 4$

and $f^3(b_k) = 1$

Notice $a_1 \in (b_1, b_2)$ and $a_2 \in (b_3, b_4)$. From the graph we deduce the graph of $f^3(x)$



From the figure we deduce $f^3(x)$ has $2^3 = 8$ fixed points of which two are $\bar{x}_1$ and $\bar{x}_2$ of $f(x)$.

The remaining 6 points must be associated with two period 3 orbits of $f(x)$. Specifically $\exists q_1, q_2$ where $q_1 \neq q_2$ s.t.

$$\gamma(q_1) = \{q_1, f(q_1), f^2(q_1), q_1, \ldots\}$$

$$\gamma(q_2) = \{q_2, f(q_2), f^2(q_2), q_2, \ldots\}$$

<u>Note</u>: Can't be 3 period 2 orbits of $f(x)$. Why?

<u>Induction</u>: $x \mapsto 4x(1-x)$ has periodic orbits of all periods

<u>Partial Summary of all Periodic orbits</u>

k	$\deg f^k(x)$	n_1	n_2	n_3	n_4	n_5
1	2	2	0	0	0	0
2	4	2	1	0	0	0
3	8	2	0	2	0	0
4	16	2	*	*	*	*
5	*	*	*	*	*	*

n_k = number of period k orbits