EXPLICIT SOLUTIONS $\quad f(x) = \mu x(1-x)$

when $\mu = 2$ the known solution to $x_{n+1} = f(x_n)$ is

$$x_n = \frac{1}{2} - \frac{1}{2}(1-2x_0)^{2^n}$$

and for $x_0 \in (0,1)$ one finds $x_n \to \frac{1}{2}$ which is the nonzero fixed point of $x \mapsto f(x)$

In the chaotic regime $\mu = 4$ the explicit solution is

$$\boxed{x_n = \sin^2(2^n \theta\pi)} \tag{1}$$

where for $x_0 \in (0,1)$

$$\theta\pi = \arcsin\left(x_0^{1/2}\right) \qquad \theta \in (0, \tfrac{1}{2})$$

<u>Theorem</u>: For $\theta \in (0,\frac{1}{2})$ one of the following is true

(a) θ is rational in which case $\{x_n\}$ is <u>eventually</u> periodic

(b) θ is irrational in which case $\{x_n\}$ is not <u>eventually</u> periodic.

Here $\{x_n\}$ is eventually periodic if $\exists N$ s.t. $\{x_n\}_{n \geq N}$ is periodic of some period.

Proof of (a) First we note f in

$$z_n = f(\theta) \equiv \sin^2(2^n \theta \pi)$$

is $\frac{1}{2}$-periodic in θ since for $n \geq 1$

$$f(\theta + \tfrac{1}{2}) = \sin^2(2^n \theta\pi + 2^{n-1}\pi) = f(\theta)$$

Next we let θ be a rational number in $(0, \frac{1}{2})$.
Thus there are integers p, q such that

$$\theta = \frac{p}{2q}$$

Moreover, $\theta \in S$ where S is the finite set

$$S = \left\{ \frac{1}{2q}, \frac{2}{2q}, \cdots \frac{p}{2q}, \cdots \frac{q-1}{2q} \right\}$$

Now, let n_1 be the smallest integer such that

$$2^{n_1}\theta = N_1 + \theta_1 \qquad \theta_1 = \frac{p_1}{2q} \in S$$

for integers N_1, p_1. In a similar fashion let $n_2 > n_1$ be the next largest integer such that

$$2^{n_2}\theta = N_2 + \theta_2 \qquad \theta_2 = \frac{p_2}{2q} \in S$$

for integers N_2, p_2. Continue this to develop a sequence of θ_K

$$\theta_1, \theta_2, \cdots \theta_K, \cdots$$

all of which are in the finite set S.

From these θ_K we construct a subsequence z_{n_K} from the orbit z_n where

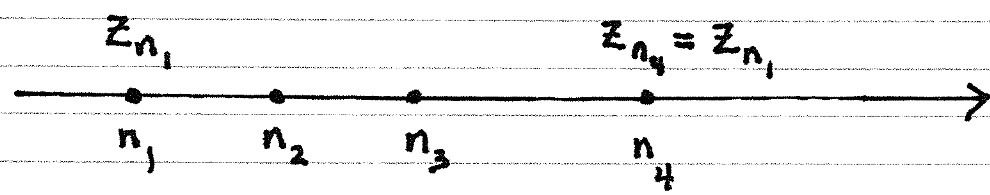
$$z_{n_K} = \sin^2\left(2^{n_K}\theta_K \pi\right)$$

$$z_{n_K} = \sin^2\left((N_K + \theta_K)\pi\right)$$

periodicity

$$z_{n_K} = \sin^2(\theta_K \pi)$$

<u>Key</u>: $\{z_{n_K}\}$ is a sequence that can only attain a finite number of values since $\{\theta_K\}$ is in S, a finite set. Thus, z_{n_K} must eventually repeat:



shows a period $T = n_4 - n_1$ orbit. $\square$

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> theta:=5/13;
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$$\theta := \frac{5}{13} \qquad (1)$$

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> for n from 1 to 12 do z[n]:=sin(2^n*theta*Pi)^2 od;
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$$z_1 := \sin\left(\frac{3}{13}\pi\right)^2$$

$$z_2 := \sin\left(\frac{6}{13}\pi\right)^2$$

$$z_3 := \sin\left(\frac{1}{13}\pi\right)^2$$

$$z_4 := \sin\left(\frac{2}{13}\pi\right)^2$$

period 6 for

$\theta = \frac{5}{13}$

$$z_5 := \sin\left(\frac{4}{13}\pi\right)^2$$

$$z_6 := \sin\left(\frac{5}{13}\pi\right)^2$$

$$z_7 := \sin\left(\frac{3}{13}\pi\right)^2$$

$$z_8 := \sin\left(\frac{6}{13}\pi\right)^2$$

$$z_9 := \sin\left(\frac{1}{13}\pi\right)^2$$

$$z_{10} := \sin\left(\frac{2}{13}\pi\right)^2$$

$$z_{11} := \sin\left(\frac{4}{13}\pi\right)^2$$

$$z_{12} := \sin\left(\frac{5}{13}\pi\right)^2 \qquad (2)$$