**Explicit Solutions** \( f(x) = \mu x (1-x) \)

When \( \mu = 2 \) the known solution to \( x_{n+1} = f(x_n) \) is

\[
x_n = \frac{1}{2} - \frac{1}{2} \left( 1 - 2x_0 \right)^{2^n}
\]

and for \( x_0 \in (0,1) \) one finds \( x_n \to \frac{1}{2} \) which is the non-zero fixed point of \( x \mapsto f(x) \)

In the chaotic regime \( \mu = 4 \) the explicit solution is

\[
(1) \quad x_n = \sin^2 \left( 2^n \theta \pi \right)
\]

where for \( x_0 \in (0,1) \)

\[
\theta \pi = \arcsin \left( x_0^{1/2} \right) \quad \theta \in (0, \frac{1}{2})
\]

**Theorem:** For \( \theta \in (0, \frac{1}{2}) \) one of the following is true

(a) \( \theta \) is rational in which case \( \{x_n\} \) is eventually periodic

(b) \( \theta \) is irrational in which case \( \{x_n\} \) is not eventually periodic.

Here \( \{x_n\} \) is eventually periodic if \( \exists N \in \mathbb{N} \) s.t. \( \{x_n\}_{n=N}^{\infty} \) is periodic of some period.
Proof of (a) First we note \( f \) in

\[ z_n = f(\theta) = \sin^2(2^n \theta \pi) \]

is \( \frac{1}{2} \)-periodic in \( \theta \) since for \( n \geq 1 \)

\[ f(\theta + \frac{\pi}{2}) = \sin^2(2^n \theta \pi + 2^{n-1} \pi) = f(\theta) \]

Next we let \( \theta \) be a rational number in \((0, \frac{1}{2})\). Thus there are integers \( p, q \) such that

\[ \theta = \frac{p}{2^q} \]

Moreover, \( \theta \in S \) where \( S \) is the finite set

\[ S = \left\{ \frac{1}{2^q}, \frac{2}{2^q}, \cdots, \frac{p}{2^q}, \cdots, \frac{q-1}{2^q} \right\} \]

Now, let \( n_1 \) be the smallest integer such that

\[ 2^{n_1} \theta = N_1 + \theta_1 \quad \theta_1 = \frac{p_1}{2^q} \in S \]

for integers \( N_1, p_1 \). In a similar fashion let \( n_2 > n_1 \) be the next largest integer such that

\[ 2^{n_2} \theta = N_2 + \theta_2 \quad \theta_2 = \frac{p_2}{2^q} \in S \]

for integers \( N_2, p_2 \). Continue this to develop a sequence of \( \theta_k \)

\[ \theta_1, \theta_2, \ldots, \theta_k, \ldots \]

all of which are in the finite set \( S \).
From these $\theta_k$ we construct a subsequence $Z_{n_k}$ from the orbit $Z_n$ where

$$Z_{n_k} = \sin^2 \left( 2^{n_k} \theta_k \pi \right)$$

$$Z_{n_k} = \sin^2 \left( (N_k + \theta_k) \pi \right) \quad \text{(periodicity)}$$

$$Z_{n_k} = \sin^2 (\theta_k \pi)$$

**Key:** $\{Z_n\}$ is a sequence that can only attain a finite number of values since $\{\theta_k\}$ is in $S$, a finite set. Thus, $Z_{n_k}$ must eventually repeat:

$$Z_{n_1} \quad Z_{n_q} = Z_{n_q}$$

$$n_1 \quad n_2 \quad n_3 \quad n_q$$

shows a period $T = n_q - n_1$ orbit. \qed
\[ \theta := \frac{5}{13} \]

\[ \text{for } n \text{ from 1 to 12 do } z[n] := \sin\left(2^n \theta \pi \right)^2 \text{ od;} \]

1. \( z_1 := \sin\left(\frac{3}{13} \pi \right)^2 \)
2. \( z_2 := \sin\left(\frac{6}{13} \pi \right)^2 \)
3. \( z_3 := \sin\left(\frac{1}{13} \pi \right)^2 \)
4. \( z_4 := \sin\left(\frac{2}{13} \pi \right)^2 \)
5. \( z_5 := \sin\left(\frac{4}{13} \pi \right)^2 \)
6. \( z_6 := \sin\left(\frac{5}{13} \pi \right)^2 \)

7. \( z_7 := \sin\left(\frac{3}{13} \pi \right)^2 \)
8. \( z_8 := \sin\left(\frac{6}{13} \pi \right)^2 \)
9. \( z_9 := \sin\left(\frac{1}{13} \pi \right)^2 \)
10. \( z_{10} := \sin\left(\frac{2}{13} \pi \right)^2 \)
11. \( z_{11} := \sin\left(\frac{4}{13} \pi \right)^2 \)
12. \( z_{12} := \sin\left(\frac{5}{13} \pi \right)^2 \)

Period 6 for

\[ \theta = \frac{5}{13} \]

(2)