Itineraries and Transition Graphs - Introduction

Recall the logistic map $x \longmapsto f(x) = 4x(1-x)$ maps the interval $[0,1]$ onto $[0,1]$. That is to say

$$f([0,1]) = [0,1]$$

As a consequence all orbits $\gamma(x_0)$ with $x_0 \in [0,1]$ remain in the interval $[0,1]$

Next we define a cover of $[0,1]$ by

$$[0,1] = A \cup B \qquad A \cap B = \phi$$

Then to each orbit

$$\gamma(x_0) = \{x_0, x_1, x_2, \ldots\}$$

we can associate an itinerary

$$I(\gamma(x_0)) = \{s_0, s_1, s_2, \ldots\}$$

of $\gamma(x_0)$ where

$$s_i = A \quad \text{if} \quad x_i \in A$$

$$s_i = B \quad \text{if} \quad x_i \in B$$

Here A and B are called symbols.

EXAMPLE logistic map $f(x) = 4x(1-x)$

Let $L = [0, \frac{1}{2})$ and $R = [\frac{1}{2}, 1]$ so that

$$[0,1] = L \cup R \qquad L \cap R = \emptyset$$

then

$$\gamma(\tfrac{1}{3}) = \{\tfrac{1}{3}, \tfrac{8}{9}, \tfrac{32}{81}, \dots\} \qquad I(\gamma(\tfrac{1}{3})) = \{L, R, L, \dots\}$$

$$\gamma(\tfrac{1}{4}) = \{\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}, \dots\} \qquad I(\gamma(\tfrac{1}{4})) = \{L, R, R, \dots\}$$

$$\gamma(\tfrac{1}{2}) = \{\tfrac{1}{2}, 1, 0, \dots\} \qquad I(\gamma(\tfrac{1}{2})) = \{R, R, L, \dots\}$$

Transition Graph Arcs

Suppose A, B are symbols for the cover of the interval $S = [0,1]$.

$$S = A \cup B \qquad A \cap B = \emptyset$$

then

$$\textcircled{A} \longrightarrow \textcircled{B}$$

if $\forall b \in B$ there exists a (preimage) $a \in A$ s.t.

$$f(a) = b$$

$$f(A) \supset B$$

The collection of all such arcs is the **transition graph**

Transition graphs for itineraries (general)

Let $f: \mathbb{R} \to \mathbb{R}$ and $f(A) = A$ where

$$A = \bigcup_{j=1}^{N} A_j \qquad \text{(cover)}$$

$$A_i \cap A_j = \emptyset \qquad i \neq j \quad \text{(disjoint)}$$

So that itineraries have N-symbols

$$\gamma(x_0) = \{x_0, x_1, x_2, \ldots\}$$

$$I(\gamma) = \{s_0, s_1, s_2, \ldots\}$$

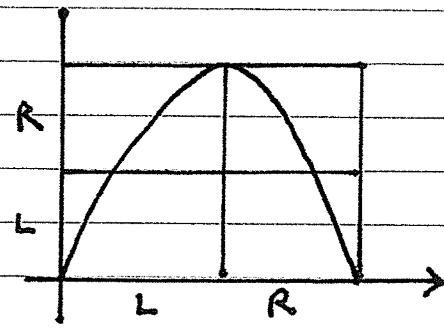
where $s_j = A_k$ if $x_j \in A_k$.

To create the transition graph having A_j as nodes we have (directed) arcs from A_i to A_j if

$$\boxed{A_i \longrightarrow A_j \qquad A_j \subset f(A_i)}$$

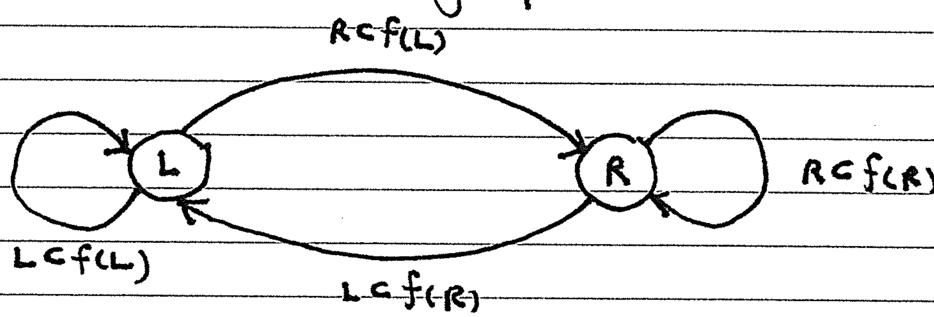
"Hence, at least some orbits landing in A_j come from A_i"

EXAMPLE $L = [0, \frac{1}{2})$ and $R = [\frac{1}{2}, 1]$ for



$f(x) = 4x(1-x)$

has the transition graph

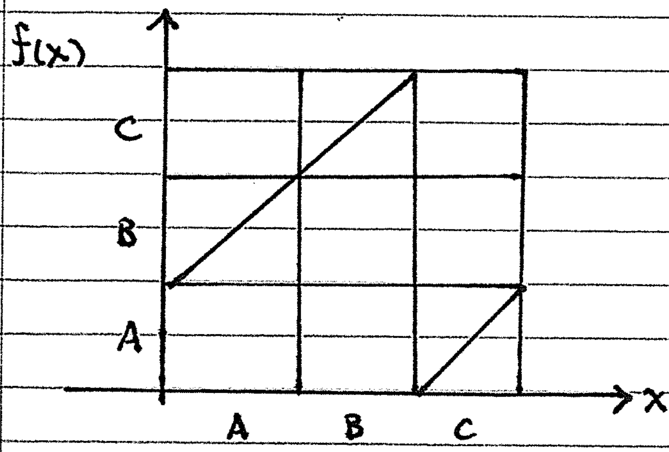


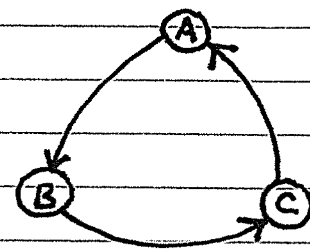
From the graph we can deduce that <u>any</u> itinerary is possible:

$$I(\gamma) = \{R, L, L, R, R, L, \ldots\}$$

This simple observations shows how non-periodic trajectories can exist since any periodic orbit must have a periodic itinerary.

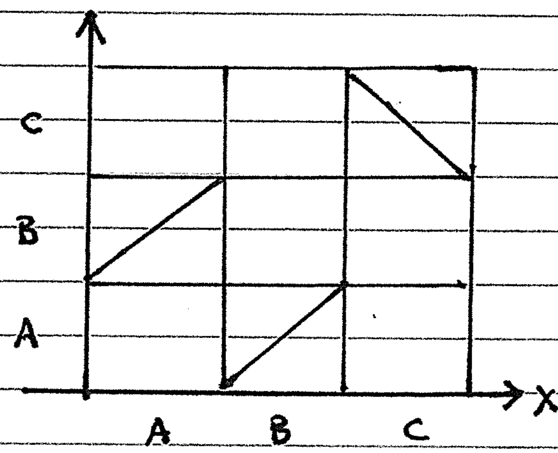
EXAMPLE Interval Exchange Map #1

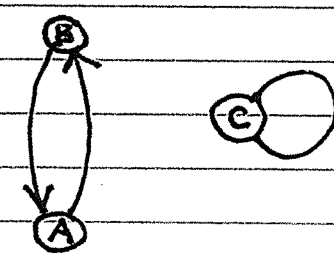




since $f(A) = B$, $f(B) = C$, $f(C) = A$.

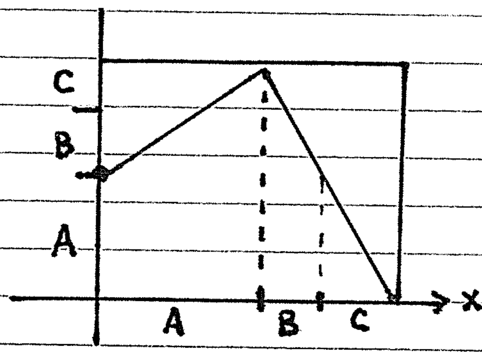
EXAMPLE Interval Exchange Map #2

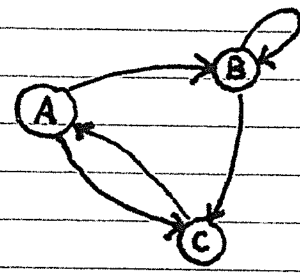




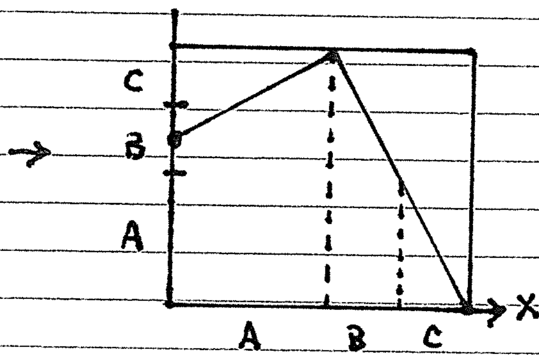
since $f(B) = A$, $f(A) = B$, $f(C) = C$

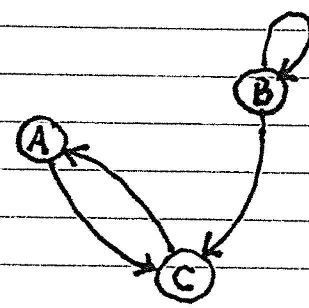
EXAMPLE Important non arc example





We alter the values of $f(x)$ on A slightly





Notice the absence an arc from A to B in the transition graph.

While it is true some of B is in A not all of B is in A

$$B \not\subset f(A)$$