Existence and uniqueness examples

EX $\qquad \dot{x} = \sqrt{x} \qquad x(0) = 0$

This IVP has two solutions

$$x_1(t) = 0 \qquad \forall t \geq 0$$

$$x_2(t) = \frac{1}{4}t^2 \qquad \forall t \geq 0$$

EX Sometimes solutions exist but not for all time t

$$\dot{x} = x^2 + 1 \qquad x(0) = 0$$

Since the equation is separable an explicit solution can be found

$$\int_0^x \frac{dz}{z^2+1} = \int_0^t dt$$

yields

$$x(t) = \tan t$$

Here the solution exhibits "blowup" at $t = \pi/2$ hence only exists on

$$t \in [0, \tfrac{\pi}{2})$$

Existence Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f = f(x,t)$ and assume $f(x,t)$ and $f_x(x,t)$ are continous on the (closed) rectangular domain

$$R = \{ (x,t) : |x - x_0| \le b \ , \ t \in [0, a] \}$$

Define

$$(1) \qquad M \equiv \max_R |f(x,t)| \qquad \alpha \equiv \min\left\{ a, \frac{b}{M} \right\}$$

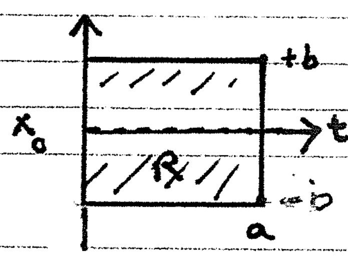
Then

$$(2) \qquad \dot{x} = f(x,t) \qquad x(0) = x_0$$

has a unique solution for $0 \le t \le \alpha$.

Pf/ Beyond scope of course.

EX $\qquad \dot{x} = x(\cos x + 3\sin x) \qquad x(0) = 0$



Shows a sample rectangle R. Need to find an upper bound for M.

$$|f(x)| \le 4|x| \le 4b$$

Since $|f(x)| \le 4b$ its true max on R has $M \le 4b$.

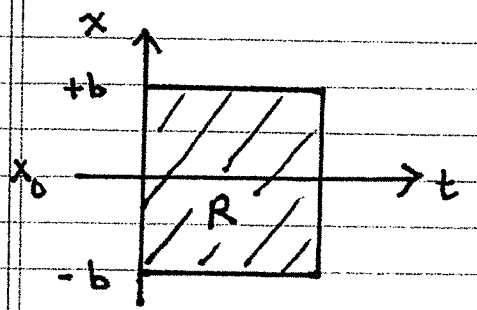
$$\alpha = \min\left\{ a, \frac{b}{M} \right\} > \min\left\{ a, \frac{1}{4} \right\}$$

We can say the IVP has a unique soln for

$$\forall t \in [0, 1/4]$$

Ex Use the existence to find a lower bound for α when

(1) $$\dot{x} = x^2 + 1 \qquad x(0) = 0$$



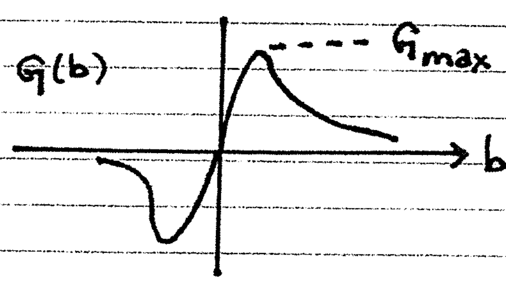
For $f(x) = x^2 + 1$ we have

$$M = \max_R f(x) = b^2 + 1$$

From the theorem

(2) $$\alpha = \min \left\{ a, \underbrace{\frac{b}{b^2+1}}_{G(b)} \right\} \qquad b > 0$$

Note in (2) one can show $G'(1) = 0$ and



$$G_{max} = G(1) = \tfrac{1}{2} = \alpha^*$$

Hence (1) has a unique soln $\forall t$:

(3) $$t \in [0, \tfrac{1}{2}]$$

<u>Remark:</u> $x(t) = \tan t$ is the solution of (1) and certainly exist for all t in (3). But note what we found in (2) was a lower bound for all t for which $x = \tan t$ exists, i.e. it exists for $t > 1/2$.

Picard iteration

(1) $\dot{x} = f(x)$ $\quad x(0) = x_0$

Integrate (1) in t to get

(2) $$x(t) = x_0 + \int_0^t f(x(s))\,ds$$

In Picard iteration one computes a sequence $\{x_n(t)\}$ where $n = 0, 1, 2 \ldots$ and $x_0(t) = x_0$ from

(3) $$\boxed{x_{n+1}(t) = x_0 + \int_0^t f(x_n(s))\,ds}$$

Picard Iteration ***
$x_n(t) \to x(t)$

Remark: To prove (1) has a unique solution one regards (3) as a fixed point problem $x = T(x)$ where T is contracting on a Banach Space. HARD!

Ex For $\dot{x} = x^2 + 1$, $x(0) = 0$ use (3) to get

$$x_1(t) = t$$

$$x_2(t) = t + \tfrac{1}{3}t^3$$

$$x_3(t) = t + \tfrac{1}{3}t^3 + \tfrac{2}{15}t^5 + \tfrac{1}{63}t^7$$

Can be contrasted with Taylor series of solution

$$x(t) = t + \tfrac{1}{3}t^3 + \tfrac{2}{15}t^5 + \tfrac{17}{315}t^7 + \cdots$$

where $x(t) = \tan t$

Taylor Series Approximations

So long as $f(x)$ is sufficiently smooth, one can find Taylor series approximations of $x(t)$

$$(1) \qquad \dot{x} = f(x(t)) \qquad x(0) = x_0$$

Such an expansion looks like

$$(2) \qquad x(t) = x(0) + \dot{x}(0)\,t + \frac{1}{2!}\ddot{x}(0)\,t^2 + \cdots$$

We differentiate (1) in t to find $\ddot{x}(0), \ldots$ using Chain rule

$$(3) \qquad \ddot{x} = f_x(x)\,\dot{x} = f_x(x)\,f(x)$$

and again

$$\dddot{x} = f_{xx}\,f\,\dot{x} + f_x\,f_x\,\dot{x}$$

$$(4) \qquad \dddot{x} = f_{xx}f^2 + f f_x^2$$

Given (1)-(4) we summarize the coefficients in the T-series (2)

$$(5) \qquad \boxed{\begin{aligned} x(0) &= x_0 \\ \dot{x}(0) &= f(x_0) \\ \ddot{x}(0) &= f(x_0)\,f_x(x_0) \\ \dddot{x}(0) &= f(x_0)^2 f_{xx}(x_0) + f(x_0)\,f_x(x_0)^2 \end{aligned}}$$

EXAMPLE

$$\dot{x} = x - x^3 \qquad \overset{x_0}{\downarrow} \; x(0) = 2$$

From this

$$f(x) = x - x^3 \qquad f(2) = -6$$

$$f_x(x) = 1 - 3x^2 \qquad f_x(2) = -11$$

$$f_{xx}(x) = -6x \qquad f_{xx}(2) = -12$$

Then, using (5)

$$x(0) = 2$$

$$\dot{x}(0) = (-6)$$

$$\ddot{x}(0) = (-6)(-11) = 66$$

$$\dddot{x}(0) = (-6)^2(-12) + (-6)(-11)^2 = -1158$$

Hence

$$x(t) = x(0) + \dot{x}(0)t + \frac{1}{2!}t^2\ddot{x}(0) + \frac{1}{3!}t^3\dddot{x}(0) + \cdots$$

$$x(t) = 2 - 6t + 33t^2 - 193t^3 + O(t^4)$$