Comparison Theorem

Let $f(x)$ and $g(y)$ be continuous and

$$g(y) \geq f(y) \qquad \forall y$$

Now let $x(t)$ and $y(t)$ be solutions of

$$\left.\begin{aligned} \dot{x} &= f(x) \qquad & x(0) &= x_0 \\ \dot{y} &= g(y) \qquad & y(0) &= x_0 \end{aligned}\right\} \text{same}$$

Then,

$$y(t) \geq x(t)$$

Pf/ $h(t) = y(t) - x(t)$ has $h(0) = 0$ and increasing since

$$\frac{dh}{dt} = g(y(t)) - f(x(t)) > 0 \qquad \square$$

EXAMPLE

$$(1) \qquad \dot{y} = y^4 + y^2 + 1 \qquad y(0) = 1$$

$$(2) \qquad \dot{x} = x^2 + 1 \qquad x(0) = 1$$

Here $g(y) \geq f(y)$ and $x(t) = \tan(t + \frac{\pi}{4})$ blows up. By the comparison theorem

$$y(t) \geq x(t)$$

hence we conclude $y(t)$ blows up at some

$$t \leq \frac{\pi}{4}$$

Potential Functions $V(x)$

$$(1) \qquad \dot{x} = f(x) = -\frac{dV}{dx}$$

where the potential function $V(x)$ is

$$V(x) = -\int^{x} f(s)\,ds$$

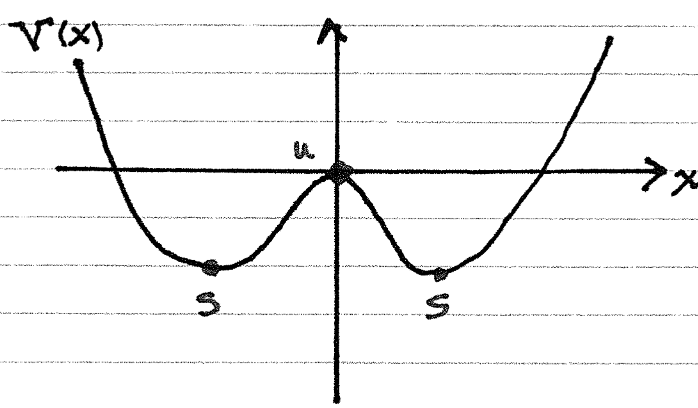
So long as $f(x)$ is continuous, (1) has a potential function. When $f(x)$ is differentiable the concavity of $V(x)$ near fixed points determine their stability. True since

$$-V''(\bar{x}) = f'(\bar{x}) > 0 \Rightarrow \text{unstable} \qquad V(x)$$

$$-V''(\bar{x}) = f'(\bar{x}) < 0 \Rightarrow \text{stable} \qquad V(x)$$

<u>EXAMPLE</u> Bistable ODE

$$\dot{x} = x - x^3 \qquad V(x) = \frac{1}{4}x^2(2 - x^2)$$



$\bar{x} = 0, \pm 1$

Here $V' = x^3 - x$ and $V'' = 3x^2 - 1$

$V''(0) < 0$ u

$V''(\pm 1) > 0$ s

In physics, V has "potential wells" and solns approach stable equilibria.