**Defn**: Let \( f: \mathbb{R} \to \mathbb{R} \). A point \( x_0 \) has sensitive dependence on initial conditions if \( \exists d > 0 \) such that for any neighbourhood \( N_r(x_0) \), \( \exists x \in N_r(x_0) \) such that

\[
|f^k(x) - f^k(x_0)| \geq d
\]

for some integer \( k \). \( x_0 \) is called a sensitive point.

Roughly speaking, \( f \) eventually maps some points near \( x_0 \) a distance \( d \) away.

is true no matter how small \( N_r(x_0) \) is.
**Example**  Tent map has sensitive points

![Graph showing a tent map](null)

with the shared itineraries:

- \[ K=1 \]
- \[ K=2 \]
- \[ K=3 \]

Note the length of all shared itinerary intervals is \(2^{-k}\). The number of such intervals is \(2^k\).

Consider intervals at \(k+2\) where \(S_j\) are symbols

\[
\begin{align*}
I_1 &= S_1 S_2 \ldots S_k LL \\
I_2 &= S_1 S_2 \ldots S_k LR \\
I_3 &= S_1 S_2 \ldots S_k RR \\
I_4 &= S_1 S_2 \ldots S_k RL
\end{align*}
\]
Pick four initial conditions \( x_i \in I_i \)

\[ \begin{array}{cccc}
  x_1 & x_2 & x_3 & x_4 \\
  \hline
  \text{LL} & \text{LR} & \text{RR} & \text{RL}
\end{array} \]

shows \( x_i \) in interval \( S_i \ldots S_k \) of length \( \frac{1}{2^k} \).
Clearly

\[ |x_i - x_j| \leq \frac{1}{2^k} \]

After \( k \) iterates \( f \) maps \( S_i \ldots S_k \) into LL, LR, RR and RL.

So for instance \( f^k(x_2) \in LR \) and \( f^k(x_4) \in RL \),

\[ |x_2 - x_4| \leq \frac{1}{2^k} \quad \text{and} \quad |f^k(x_2) - f^k(x_4)| \geq \frac{1}{4} = d \]

since points in LR and RL are at least \( \frac{1}{4} \) apart.

For example if \( k = 1000 \) we can find 2 initial conditions in \( S_i \ldots S_{1000} \) (whose length is \( 2^{-1000} \approx 10^{-300} \)) such that after 1000 iterates their images are \( \frac{1}{4} \) apart.
Liapunov Exponents

Let \( f: \mathbb{R} \to \mathbb{R} \) have a fixed point \( \bar{x} \) and

\[
(1) \quad x_{n+1} = f(x_n)
\]

Next we define the distance \( y_n \) of \( x_n \) from \( \bar{x} \) at the \( n^{th} \) iterate

\[
(2) \quad y_n = x_n - \bar{x}
\]

Here we presume \( y_n \) is small in the following Taylor series approximation

\[
x_{n+1} - \bar{x} = f(\bar{x} + y_n) - \bar{x}
\]

\[
y_{n+1} = f(\bar{x}) - \bar{x} + f'(\bar{x})y_n + O(y_n^2)
\]

Concluding

\[
y_{n+1} = f'(\bar{x})y_n + O(y_n^2)
\]

So when \( \bar{x} \) is unstable, the distance \( y_n \) grows by a (multiplier) factor of

\[
\lambda = |f'(\bar{x})| > 1
\]

in each iterate.
Now suppose

\[ y(p_i) = \{p_1, p_2, \ldots, p_k, p_1, p_2, \ldots \} \]

is a period \( k \) orbit of \( f \). The stability is determined by

\[ (3) \quad \lambda_k = \left| \frac{d}{dx} f^k(x) \right|_{x=p_j} \]

since then \( p_j \) is a stable or unstable fixed point of the \( k^{th} \) iterate map \( x \mapsto f^k(x) \).

The multiplier \( \lambda_k \) in (3) can be computed by an induction argument (do in class) and is

\[ (4) \quad \lambda_k = \left| f'(p_1) f'(p_2) \cdots f'(p_k) \right| \]

One may use the product symbol to express \( \lambda_k \)

\[ \lambda_k = \prod_{j=1}^{k} |f'(p_j)| \]

Thus the distance \( y_{n+k} = \lambda_k y_n \) after \( k \)-iterates.
On average the amplification of the distance per iterate is

\[ \lambda = \left| f'(p_1) f'(p_2) \cdots f'(p_k) \right|^{\frac{1}{k}} \]

so that \( \lambda_k = \lambda^k \). We generalize this to any orbit as follows:

**Defn**: Let \( f : \mathbb{R} \to \mathbb{R} \). The **Liapunov number** of orbit \( \gamma(x_1) \) is

\[ \lambda_\infty = \lim_{n \to \infty} \left( |f'(x_1)| |f'(x_2)| \cdots |f'(x_n)| \right)^{\frac{1}{n}} \]

when it exists. The **Liapunov exponent**

\[ l = \lim_{n \to \infty} \frac{1}{n} \left( \ln |f'(x_1)| + \ln |f'(x_2)| + \cdots + \ln |f'(x_n)| \right) \]

when it exists. When both limits exist

\[ l = \ln \lambda_\infty \]
Remarks

(1) If at any \( k \) we have \( f'(x_k) = 0 \) we would have \( \lambda_\infty = 0 \) while the Liapunov exponent would be undefined since then \( \ln|f'(x_k)| \) is undefined.

This would be true for any orbit \( \gamma(x_0) \) of \( f(x) = 4x(1-x) \) eventually maps to \( x_k = \frac{1}{2} \).

Also the Liapunov exponent may be undefined if its associated series diverges.

(2) Meaning:

\[ \lambda_\infty > 1 \Rightarrow \text{trajectories eventually separate} \]

\[ L = \ln(\lambda_\infty) > 0 \Rightarrow \text{trajectories eventually separate} \]
Defn: An orbit \( \gamma(x_1) \) of \( f(x) \) is asymptotically periodic if there exists a period \( k \) orbit

\[ \gamma_k = \{ p_1, p_2, \ldots, p_k, \ldots \} = \{ y_1, \ldots \} \]

such that

\[ \lim_{n \to \infty} |x_n - y_n| = 0 \]

Note that if \( \gamma(x_1) \) is eventually periodic it (by defn) equals a periodic orbit. For logistic map \( \gamma(x_1) = \{ \frac{3}{2}, 1, 0, 0, \ldots \} \) is eventually periodic to the period-1 orbit \( \{ 0, 0, \ldots \} \)

We do not prove the following:

Theorem If \( \gamma(x_1) \) is asymptotically periodic to a periodic orbit \( \gamma(p_1) \) and \( f'(x_i) \neq 0 \) \( \forall i \) then \( \gamma(x_1) \) and \( \gamma(p_1) \) have the same Liapunov exponents (when they exist)
**Defn:** Let \( \gamma(x_1) \) be any bounded orbit of \( f(x) \). The orbit \( \gamma(x_1) \) is chaotic if

(i) \( \gamma(x_1) \) is not asymptotically periodic

(ii) the Liapunov \( L = \ln \lambda_\infty > 0 \)

**EXAMPLE** The Tent map has chaotic orbits

\[
\begin{align*}
&f(x) \\
&L \quad \frac{1}{2} \quad R \quad 1
\end{align*}
\]

The transition graph implies \( f(x) \) has many orbits with nonperiodic itineraries and are not asymptotic to any periodic orbit. Let \( \gamma(x_1) = \{x_1, x_2, \ldots \} \) be any such orbit.

\[
L = \lim_{n \to \infty} \frac{1}{n} \left( \frac{1}{2} \left( \ln|f'(x_1)| + \cdots + \ln|f'(x_n)| \right) \right) = \ln 2 > 0
\]