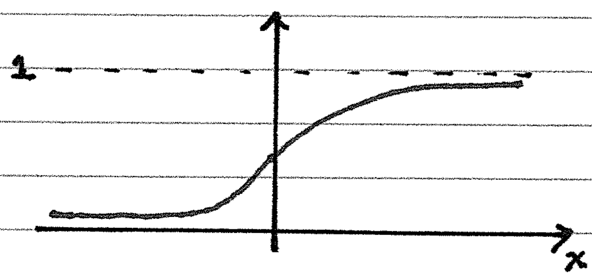
Conjugate Maps

For this discussion we will restrict our attention only to maps $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$, and possibly $f:A\to A$ where A is some interval.

<u>Defn</u>: $H:\mathbb{R}\to\mathbb{R}$ is a <u>homeomorphism</u> on $\mathbb{R}$ if it is continuous and invertible.

<u>Ex</u>: $H(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x$, $H:\mathbb{R}\to(0,1)$

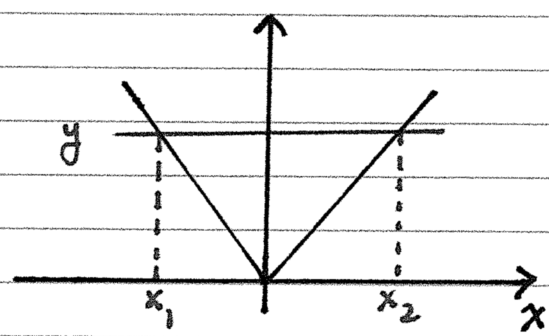


Solve $y=H(x)$ for x:

$H^{-1}(y) = \tan(\pi(y-\frac{1}{2}))$

is a homeomorphism

<u>Ex</u> $H(x) = |x|$ $H:\mathbb{R}\to\mathbb{R}^+$



(not homeomorphism)

while H is continuous it is not invertible. $\forall y>0$ the soln to

$$y = H(x)$$

is not unique. There is no <u>unique</u> x s.t $H(x)=y$.

<u>Definition:</u> The maps $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are <u>conjugate</u> on $\mathbb{R}$ if there is a homeomorphism $H(x)$ such that

$$(1) \qquad H(f(x)) = g(H(x)) \qquad \forall x \in \mathbb{R}$$

Using composition notation $\circ$, (1) is also

$$(2) \qquad H \circ f = g \circ H$$

and since H has an inverse, (2) can be written

$$(3) \qquad f = H^{-1} \circ g \circ H$$

Associated with the conjugacy is a <u>commutation diagram</u>

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "H"'] & A \arrow[d, "H"] \\ B \arrow[r, "g"] & B \end{tikzcd}$$

<u>Remark:</u> Conjugacy defines an <u>equivalence relation</u> between classes of conjugate maps.

EXAMPLE Define the maps on $[0,1]$

$$f(x) = 2x\sqrt{1-x^2}$$

$$g(x) = 4x(1-x)$$

Show these maps are conjugate on $[0,1]$ under the homeomorphism

$$H(x) = x^2$$

Note: On $[0,1]$, $H^{-1}(x) = \sqrt{x}$ is uniquely defined.

Must show $H(f(x)) = g(H(x)) \quad \forall x \in [0,1]$

$$H(f(x)) = \left(2x(1-x^2)^{1/2}\right)^2 = 4x^2(1-x^2)$$

same.

$$g(H(x)) = 4x^2(1-x^2)$$

Clearly equal.

EXAMPLE Consider the logistic/quadratic maps

$$f(x) = 4x(1-x)$$

$$g(x) = x^2 + c$$

Show these maps are conjugate under

$$H(x) = ax + b$$

for some constants a, b and c. Must show $P = 0$.

$$P(x) = H(f(x)) - g(H(x))$$

$$P(x) = af(x) + b - (ax+b)^2 + c$$

$$P(x) = \underline{-a(a+4)}\,x^2 + \underline{a(4-2b)}\,x + \underline{(b-b^2-c)}$$

Now we see $P(x) = 0$ only if underlined coefficients vanish. Hence $a = -4$ for x^2 coefficient and $b = 2$ for x coefficient. Likewise $c = -2$. Conclude:

$$f(x) = 4x(1-x)$$

is conjugate to

$$g(x) = x^2 - 2$$

under the homeomorphism $H(x) = -4x + 2$.

Conjugate maps: consequences

If maps $f(x)$ and $g(x)$ are conjugate (on A) then there is a 1-1 correspondence between orbits of $f(x)$ and orbits of $g(x)$. So, for instance, if $f(x)$ has a period 3 orbit then so does $g(x)$.

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "H"'] & \bullet \arrow[r, "f"] \arrow[d, "H"'] & \bullet \arrow[r, "f"] \arrow[d, "H"'] & \bullet \arrow[d, "H"'] & \cdots \\
\bullet \arrow[r, "g"'] & \bullet \arrow[r, "g"'] & \bullet \arrow[r, "g"'] & \bullet &
\end{tikzcd}$$

Since f and g are conjugate

$$f(x) = (H^{-1} g\, H)(x)$$

and thus

$$f^2 = H^{-1} g\, \underline{\underline{H H^{-1}}}\, g\, H = H^{-1} g^2 H$$

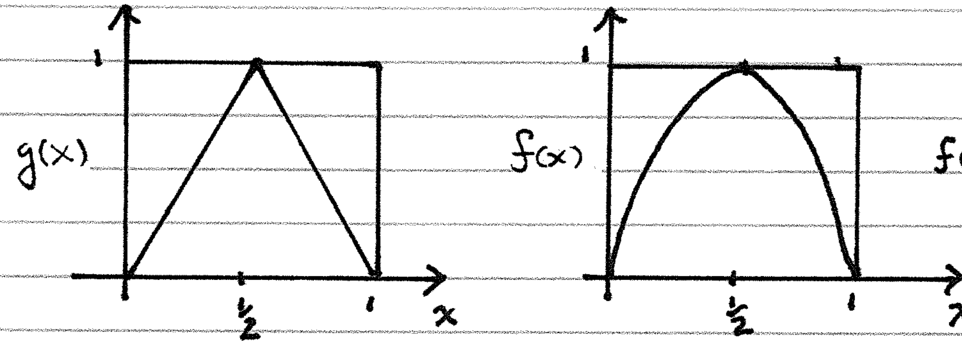
$$f^n = H^{-1} g^n H$$

So if $\gamma_f(x_0) = \{x_0, x_1, \ldots\}$ an orbit of f the associated orbit of g is

$$\gamma_g(y_0) = \{ H(x_0), H(x_1), H(x_2), \ldots \}$$

where $y_0 = H(x_0)$.

EXAMPLE Tent and Logistic conjugacy



$f(x) = 4x(1-x)$

Claim these are conjugate with

(1) $$(f \circ H)(x) = (H \circ g)(x)$$

(2) $$H(x) = \frac{1}{2}(1 - \cos \pi x)$$

Here we show the left and right sides of (1) are equal for $x \in [0, \frac{1}{2})$ where $g(x) = 2x$

$$\begin{aligned}(f \circ H)(x) &= 4\,H(x)(1 - H(x)) \\ &= 4 \cdot \frac{1}{2}(1 - \cos \pi x) \cdot \frac{1}{2}(1 + \cos \pi x) \\ &= 1 - \cos^2 \pi x = \sin^2 \pi x \quad ✓✓\end{aligned}$$

and for the right side of (1)

$$(H \circ g)(x) = H(2x) = \frac{1}{2}(1 - \cos 2\pi x) = \sin^2 \pi x \quad ✓✓$$

□