Period 3 implies chaos

In 1975, Li and Yorke proved that if a continuous map $f: A \to A$ has a period 3 orbit then

(i) it has no asymptotically stable periodic orbits

(ii) it has sensitive points

In this sense the map is chaotic

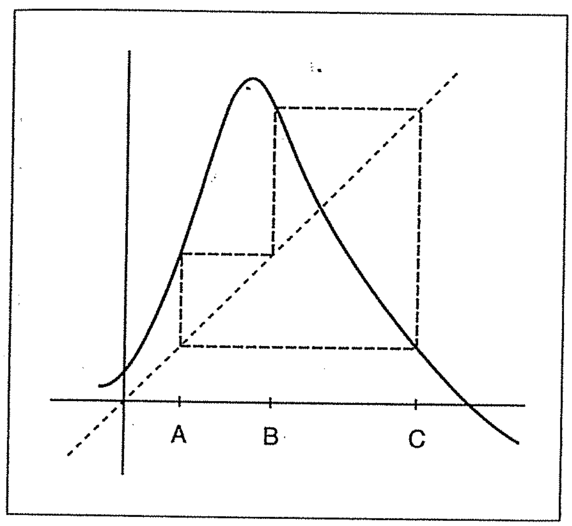


Figure 1.14 A map with a period-three orbit.
The dashed lines follow a cobweb orbit, from *A* to *B* to *C* to *A*.

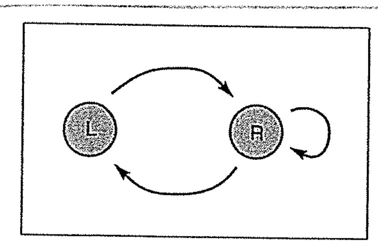
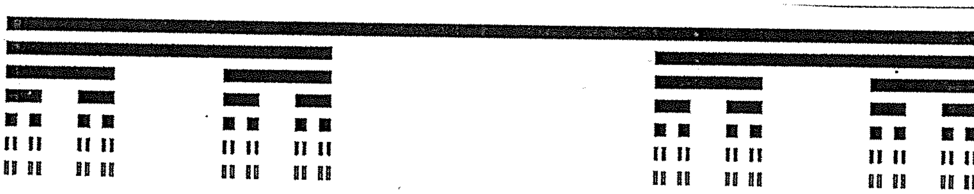


Figure 1.16 Transition graph for map with period-three orbit.
The three arrows imply that $f(\mathrm{L}) \supseteq \mathrm{R}$, $f(\mathrm{R}) \supseteq \mathrm{R}$, and $f(\mathrm{R}) \supseteq \mathrm{L}$.

Cantor sets

There are many types of Cantor sets. Here we describe the "middle thirds" Cantor set C. It is defined via an iterative process starting with $C_0 = [0, 1]$. Then C_1 is obtained by removing the middle third of C_0, i.e $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. C_2 is found by removing the middle thirds of all interals in C_1. Below shows the process



$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

Fun facts:

(1) C is uncountable

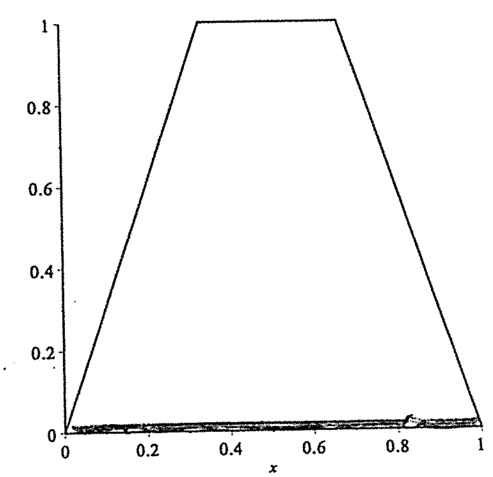
(2) C has no length.

Cantor sets from maps

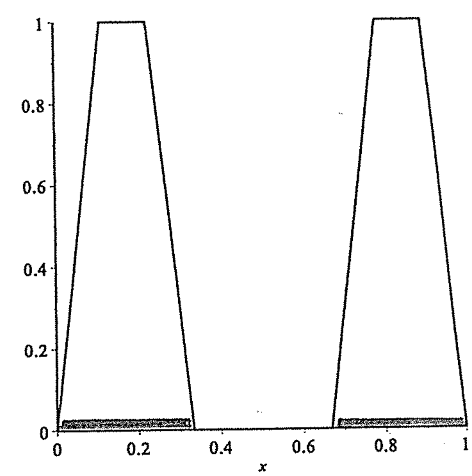
$$f(x) \equiv \begin{cases} 3x & x \in [0, \tfrac{1}{3}] \\ 1 & x \in (\tfrac{1}{3}, \tfrac{2}{3}) \\ 3 - 3x & x \in [\tfrac{2}{3}, 1] \end{cases}$$

Below are several iterates of f and the developing Cantor set:

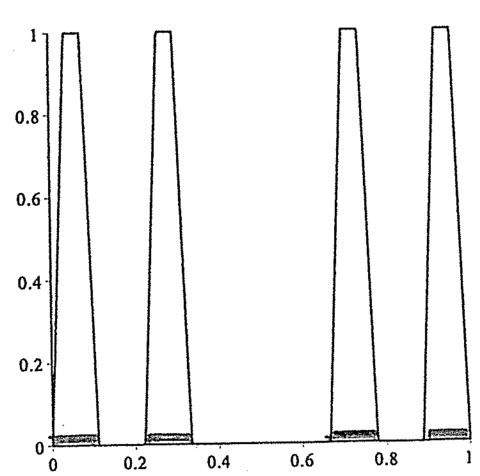
$f(x)$

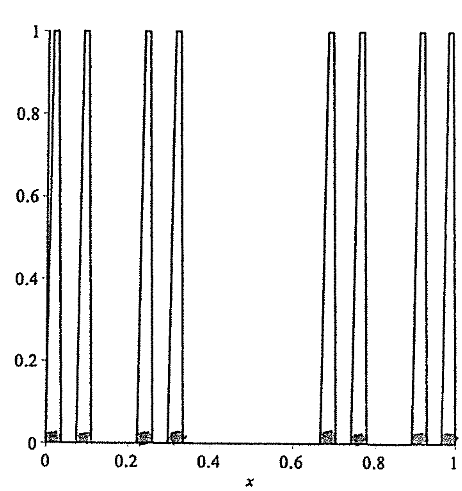


$f^2(x)$



$f^3(x)$





Binary numbers and the bit-shift map

Any $x \in [0,1]$ has a binary representation

$$x = 0.b_1 b_2 b_3 \ldots\ldots = \sum_{K=1}^{\infty} \frac{b_K}{2^K}$$

where $b_K \in \{0, 1\}$. A fact we do not prove here is that every eventually periodic sequence $\{b_n\}$ is in a 1-1 correspondence with rational numbers on $[0,1]$. Every non-eventually periodic $\{b_n\}$ correspond to irrationals on $[0,1]$.

EXAMPLES

(1) $x = \frac{1}{5} = .00110011\overline{0011}$

(2) $x = \frac{4}{7} = 0.100\overline{100}$

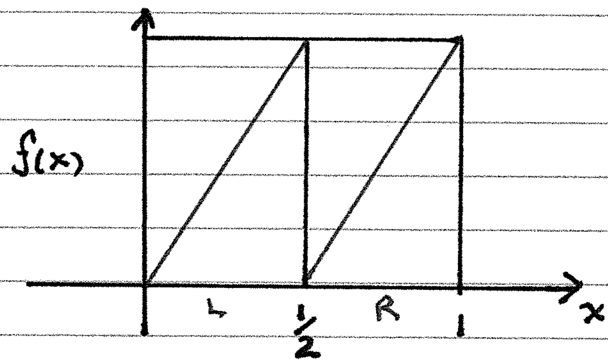
To show (2) we use the geometric series summation formula.

$$x = \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^7} + \cdots = \sum_{K=0}^{\infty} \frac{1}{2^{3K+1}}$$

$$x = \frac{1}{2} \sum_{K=0}^{\infty} \frac{1}{2^{3K}} = \frac{1}{2} \sum_{K=0}^{\infty} \left(\frac{1}{8}\right)^K \qquad \left(r = \frac{1}{8} \text{ converge}\right)$$

$$x = \frac{1}{2} \frac{1}{\left(1 - \frac{1}{8}\right)} = \frac{4}{7}$$

Bit-shift $2x \bmod 1$



$$f(x) = \begin{cases} 2x & 0 \le x < \frac{1}{2} \\ 2x-1 & \frac{1}{2} \le x \le 1 \end{cases}$$

Now, for any $x_L \in L$, $b_1 \equiv 0$ so that

$$f(x_L) = 0.b_1 b_2 b_3 \ldots\ldots$$

is a shift (left) of the binary sequence $\{b_n\}$.
Similarly, for any $x_R \in R$, $b_1 = 1$ and

$$x_R = 0.1 b_2 b_3 b_4 \ldots\ldots$$

$$2x_R = 1.b_2 b_3 b_4 \ldots\ldots$$

$$2x_R - 1 = 0.b_2 b_3 b_4 \ldots\ldots$$

We conclude $\forall x \in [0,1]$, f is the left shift of the binary representation of x

$$f(x) = 0.b_2 b_3 b_4 \ldots\ldots$$

EXAMPLE OF A PERIOD 3 ORBIT

$.110110110 \rightarrow .101101101 \rightarrow .011011011 \rightarrow .110110110$

When converted to fractional form

$$\tfrac{1}{7} \longrightarrow \frac{2}{7} \longrightarrow \frac{4}{7} \longrightarrow \frac{1}{7}$$

This alone implies $f(x)$ has chaotic orbits.

Conjugate to logistic map

$$x_{n+1} = f(x_n)$$

is conjugate to

$$y_{n+1} = 4y_n(1-y_n)$$

via the transformation

$$y_n = \sin^2(2\pi x_n)$$