Planar Maps $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$\vec{x}_{n+1} = f(\vec{x}_n)$$

defines orbits $\gamma(\vec{x}_0) = \{\vec{x}_0, \vec{x}_1, \vec{x}_2, \dots\}$

Some relevant notations

$$f(\vec{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Definition A neighbourhood $N_r(\vec{x}_0)$ of $\vec{x}_0$

$$N_r(\vec{x}_0) = \{\vec{x} : \|\vec{x} - \vec{x}_0\| < r\}$$

where the Euclidean metric

$$\|\vec{x} - \vec{x}_0\| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

So defined $N_r(\vec{x}_0)$ is a disk centered at $\vec{x}_0$ having radius r.

Definition A fixed point $\vec{z}$ of f is any point such that

$$\vec{z} = f(\vec{z})$$

EXAMPLE Orbits are sequences of points in $\mathbb{R}^2$

$$\vec{x}_{n+1} = f(\vec{x}_n) = \begin{pmatrix} 1 - y_n \\ x_n^2 \end{pmatrix} \qquad \vec{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\vec{x}_1 = \begin{pmatrix} 1-2 \\ 1^2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{x}_2 = \begin{pmatrix} 1-1 \\ (-1)^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Hence the first few terms of the orbit are

$$\gamma(\vec{x}_0) = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ldots \right\} \text{ periodic?}$$

EXAMPLE Computing fixed points $\vec{z}$ of

$$f(\vec{x}) = \begin{pmatrix} 1 - x \\ y^2 \end{pmatrix}$$

All fixed points of f solve

$$1 - x = x \qquad x = 1/2$$
$$y^2 = y \qquad y = 0, 1$$

Conclude

$$\vec{z}_1 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \qquad \vec{z}_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

EXAMPLE Are more complex than maps on $\mathbb{R}$.

$$\vec{x}_{n+1} = f(\vec{x}_n) \qquad \vec{x}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

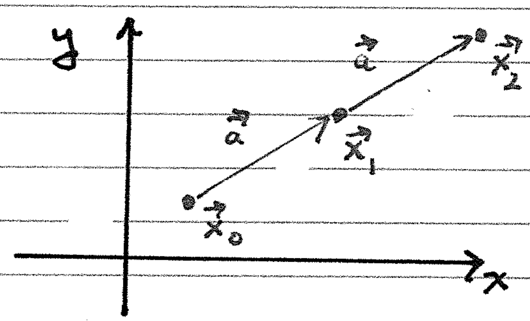
where, for example, we define

$$f(\vec{x}) = \begin{pmatrix} 4x(1-x) \\ \frac{1}{2} y \end{pmatrix}$$

then $y_n \to 0$ and $\{x_n\}$ is any orbit of the logistic map!

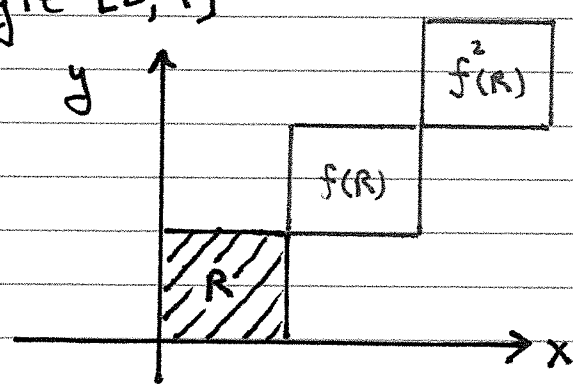
EXAMPLE Translation map

$$\vec{x}_{n+1} = \vec{x}_n + \vec{a} = f(\vec{x}_n)$$



$$\vec{x}_n = \vec{x}_0 + n\vec{a}$$

Shows the translation of $\vec{x}_0$ under f.
Suppose $\vec{a} = (1,1)$ and R is the unit rectangle $[0,1]^2$



Linear map $f(\vec{x}) = A\vec{x}$, $A \in \mathbb{R}^{2\times 2}$

Such maps satisfy the linearity property

$$f(\alpha\vec{x} + \beta\vec{y}) = \alpha f(\vec{x}) + \beta f(\vec{y})$$

$\forall \alpha, \beta \in \mathbb{R}$ and vectors $\vec{x}, \vec{y} \in \mathbb{R}^2$.

Scaling Map The simplest linear map can be defined by a diagonal A

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \qquad f(\vec{x}) = A\vec{x}$$

It is easy to show the solution of

(1) $$\vec{x}_{n+1} = A\vec{x}_n$$

is given by

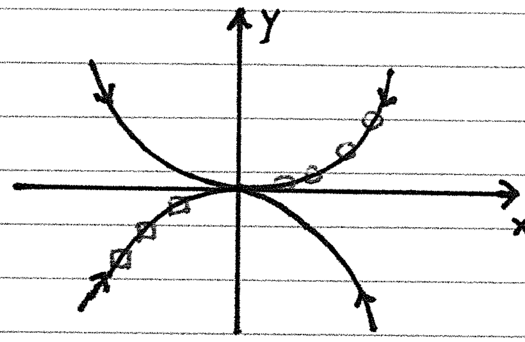
(2) $$\vec{x}_n = A^n \vec{x}_0$$

where, since A is diagonal,

(3) $$A^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

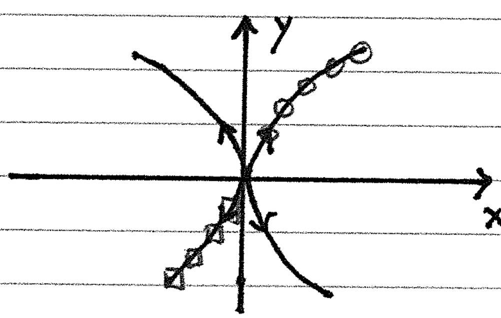
Below, orbits are given by dots. Curves drawn in are for visual purposes only.

Case: $0 < b < a < 1$ Stable sink



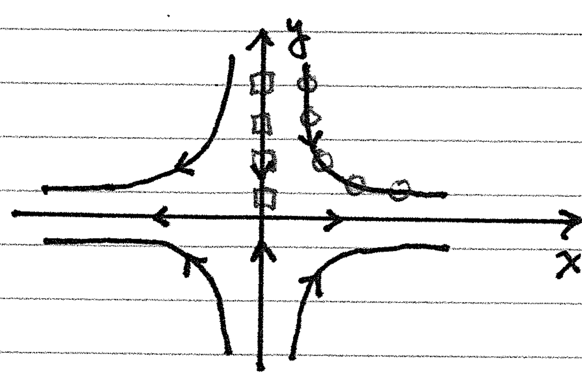
$\vec{x}_n \to 0, \quad \forall \vec{x}_0$

Case: $a > b > 1$ unstable source



$\vec{x}_n \to \infty, \quad \forall \vec{x}_0 \neq 0$

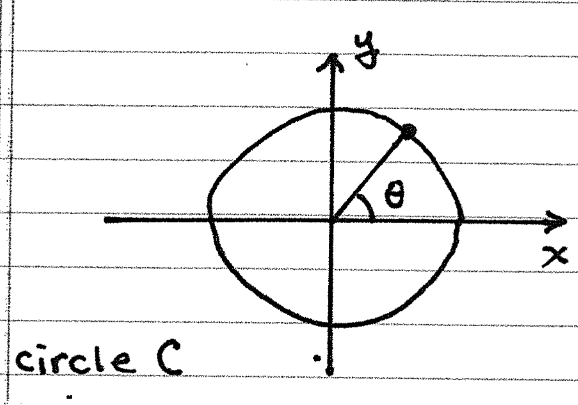
Case: $a > 1 > b > 0$ Saddle



$\vec{x}_n \to 0, \quad \forall \vec{x}_0 = \begin{pmatrix} 0 \\ y_0 \end{pmatrix}$

is a stable "manifold".

EXAMPLE Images of circles in $\mathbb{R}^2$ under scaling map in the non-saddle case $a > b > 1$



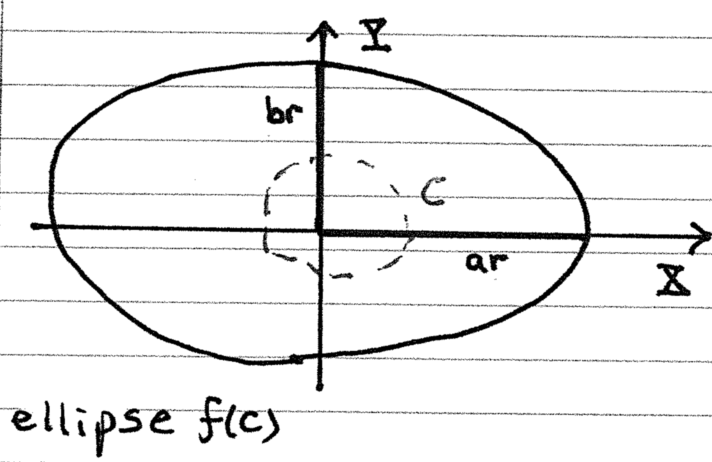
Every point can be given by the parametrization

$$\vec{x} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

$$\Big\downarrow f$$

$$\vec{X} = A\vec{x}$$

$$\vec{X} = \begin{pmatrix} ar\cos\theta \\ br\sin\theta \end{pmatrix}$$



Hence the image $f(C)$ is an ellipse with eqn

$$\left(\frac{X}{ar}\right)^2 + \left(\frac{Y}{br}\right)^2 = 1$$

For $a > b > 1$ the ellipse cointains C. Upon iterates the dimensions of $f^n(C)$ grow but remains an ellipses.