Area preserving maps

Consider an invertible coordinate transformation

\[ \overline{X} = f_1(x, y) \]
\[ \overline{Y} = f_2(x, y) \]

In vector form this defines an invertible map \( \overline{X} \mapsto f(\overline{X}) \)

\[ \overline{X} = f(\overline{X}) \]

Next consider the image \( f(R) \) of region \( R \) in the \( \overline{X} \)-plane

A result from multivariate calculus:

\[
\iint_R G(f(\overline{X})) |\det Df| d\overline{X} = \iint_{f(R)} G(\overline{X}) d\overline{X}
\]

for any continuous \( G : \mathbb{R}^2 \to \mathbb{R}^2 \).
Letting $G(x) \equiv 1$ in equation (1) yields

$$\mu(f(R)) = \int \int_{f(R)} d\mathbf{x} = \int \int_{R} \det Df \, d\mathbf{x}$$

from which we have the following

**Theorem:** If $f: \mathbb{R}^2 \to \mathbb{R}^2$ is $C^1(\mathbb{R}^2)$ and is invertible and $\det Df$ on $\mathbb{R}^2$, then $f$ is area preserving:

$$\mu(R) = \mu(f(R)) \quad \forall R \subset \mathbb{R}^2$$

**Examples:** Linear maps $Df(x) = A$ for $f(x) = Ax$

(a) $A = \begin{bmatrix} \frac{3}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{3}{4} \end{bmatrix}$ has $\lambda_1 = 2$, $\lambda_2 = \frac{1}{2}$ and $\det A = 1$ (saddle)

Since $\det A \neq 0$ it is invertible and hence area preserving by the Theorem.

(b) $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ defines counterclockwise rotation. Clearly, $\det A = 1$ so $f(x) = Ax$ is area preserving.
**Example**  

**Henon Map**

\[
\begin{align*}
x_{n+1} &= 1 - ax_n^2 + y_n \\
y_{n+1} &= bx_n
\end{align*}
\]

Thus,

\[
f(x^*) = \begin{pmatrix} 1 - ax^2 + y^* \\ bx^* \end{pmatrix}
\]

we show \( f \) is invertible by computing \( f^{-1} \).

To do so solve \( x^* = f(x^*) \) for \( x^* \):

\[
\begin{align*}
x^* &= 1 - ax^2 + y^* \\
y^* &= bx^*
\end{align*}
\]

yields

\[
f^{-1}(x^*) = \begin{pmatrix} b^{-1} y^* \\ x^* + \frac{a}{b^2} y^2 - 1 \end{pmatrix}
\]

Since

\[
Df(x^*) = \begin{bmatrix} -2ax & 1 \\ b & 0 \end{bmatrix} \quad \text{det } Df = -b
\]

we conclude the Henon map is area preserving if \( b = \pm 1 \).
**Area preserving linear maps**

Let $f(x) = A \cdot x$. If $|\det A| = 1$ the map is area preserving.

If the eigenvalues of $A$ are $\lambda_1$, $\lambda_2$ recall

\[ \det A = \lambda_1 \lambda_2 \]

\[ \text{Tr } A = \lambda_1 + \lambda_2 \]

Thus, $f$ is area preserving if $\lambda_1 \lambda_2 = 1$.

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**Diagram:**

- **Hyperbolic case:** $1 < 1 < 12_{2,1}$
  - $\lambda_1, \lambda_2$ are distinct

- **Parabolic case (repeated):** $12_{1,1} = 12_{2,1} = 1$
  - $\lambda_1 = \lambda_2$

- **Elliptic case:** $12_{k,1} = 1$
  - Conjugates $\lambda_1 = \overline{\lambda}_2$

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**Notes:**

- $12_{1,1}$ and $12_{2,1}$ denote eigenvalues.
- $\overline{\lambda}_2$ denotes the complex conjugate of $\lambda_2$. 