Area preserving maps

Consider an invertible coordinate transformation

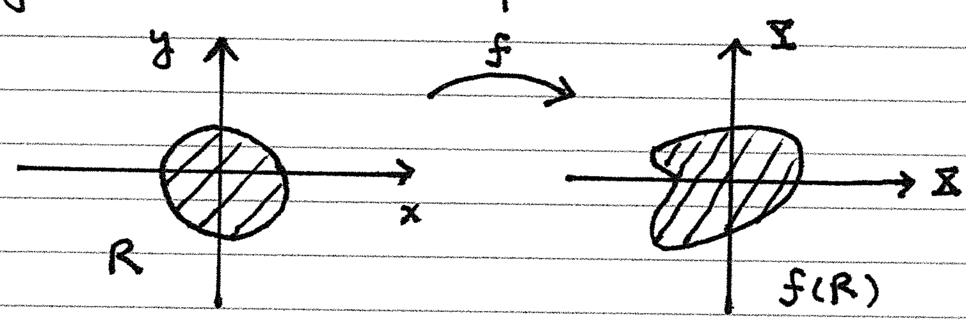
$$\bar{X} = f_1(x, y)$$

$$\bar{Y} = f_2(x, y)$$

In vector form this defines an invertible map $\vec{x} \mapsto f(\vec{x})$

$$\vec{\bar{X}} = f(\vec{x})$$

Next consider the image $f(R)$ of region R in the $\vec{x}$-plane



A result from multivariate calculus:

$$(1) \qquad \iint_R G(f(\vec{x}))\,|\det Df|\,d\vec{x} = \iint_{f(R)} G(\vec{\bar{X}})\,d\vec{\bar{X}}$$

for any continuous $G: \mathbb{R}^2 \to \mathbb{R}^2$.

Letting $G(\vec{x}) \equiv 1$ in equation (1) yields

$$(2) \qquad \mu(f(R)) = \iint_{f(R)} d\vec{x} = \iint_{R} |\det Df|\, d\vec{x}$$

from which we have the following

<u>Theorem</u>: If $f: \mathbb{R}^2 \to \mathbb{R}^2$ is $C^1(\mathbb{R}^2)$ and is invertible and $|\det Df|$ on $\mathbb{R}^2$, then f is area preserving:

$$\mu(R) = \mu(f(R)) \qquad \forall R \subset \mathbb{R}^2$$

EXAMPLES: Linear maps $Df(\vec{x}) = A$ for $f(\vec{x}) = A\vec{x}$

(a) $A = \begin{bmatrix} 5/4 & -\frac{3}{4} \\ -\frac{3}{4} & 5/4 \end{bmatrix}$ has $\lambda_1 = 2$, $\lambda_2 = \frac{1}{2}$ and $\det A = 1$ (saddle)

Since $\det A \neq 0$ it is invertible and hence area preserving by the Theorem.

(b) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ defines counterclockwise

rotation. Clearly, $\det A = 1$ so $f(\vec{x}) = A\vec{x}$ is area preserving.

EXAMPLE Henon Map

$$x_{n+1} = 1 - a x_n^2 + y_n$$
$$y_{n+1} = b x_n$$

Thus,

$$f(\vec{x}) = \begin{pmatrix} 1 - a x^2 + y \\ b x \end{pmatrix}$$

we show f is invertible by computing f^{-1}.
To do so solve $\vec{X} = f(\vec{x})$ for $\vec{x}$:

$$X = 1 - a x^2 + y$$
$$Y = b x$$

yields

$$f^{-1}(\vec{X}) = \begin{pmatrix} b^{-1} Y \\ X + \frac{a}{b^2} Y^2 - 1 \end{pmatrix}$$

Since

$$Df(\vec{x}) = \begin{bmatrix} -2ax & 1 \\ b & 0 \end{bmatrix} \qquad \det Df = -b$$

we conclude the Henon map is area preserving if $b = \pm 1$.

Area preserving linear maps

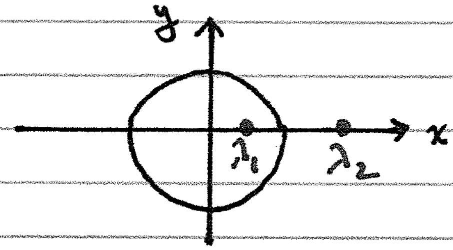
Let $f(\vec{x}) = A\vec{x}$. If $|\det A| = 1$ the map is area preserving.

If the eigenvalues of A are λ_1, λ_2 recall

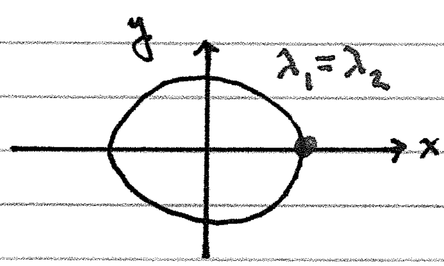
$$\det A = \lambda_1 \lambda_2$$

$$\operatorname{Tr} A = \lambda_1 + \lambda_2$$

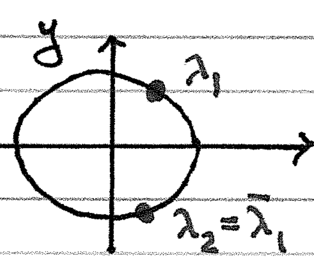
Thus, f is area preserving if $\lambda_1 \lambda_2 = 1$.



$|\lambda_1| < 1 < |\lambda_2|$
hyperbolic case



$|\lambda_1| = |\lambda_2| = 1$
parabolic cape
(repeated)



$|\lambda_k| = 1$ conjugates
elliptic case

$\lambda_1 = \bar{\lambda}_2$