Complex Eigenvalue case

Recall that for complex λ_i we have

$$(1) \qquad f(\vec{x}) = P C P^{-1} \vec{x}$$

where P is real and for $\lambda = a \pm ib$; $a, b \in \mathbb{R}$,

$$(2) \qquad C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

If we define $g(\vec{y}) = C\vec{y}$ then (1) implies f is conjugate to g

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "P^{-1}"] & \cdot \arrow[d, "P^{-1}"] \\ y \arrow[r, "g"] & \cdot \end{tikzcd}$$

So, wlog we consider

$$(3) \qquad f(\vec{x}) = C \vec{x}$$

In polar coordinates

$$(4) \qquad \lambda = |\lambda| (\cos\theta + i \sin\theta) = |\lambda| e^{i\theta}$$

Given (2)-(3) we may now write (3) as

$$f(\vec{x}) = |\lambda| \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

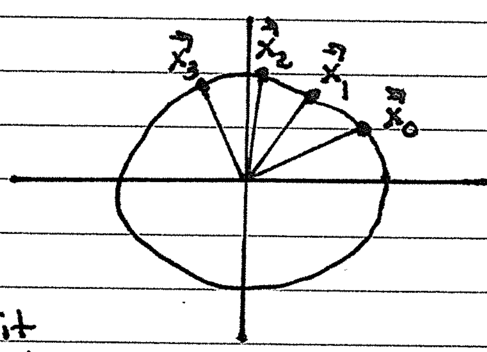
$$f(\vec{x}) = |\lambda| \quad R(\theta)$$

where $R(\theta)$ is the (counterclockwise) rotation matrix. Consequently, orbits of $f(\vec{x})$ are given by

$$\vec{x}_n = \underbrace{|\lambda|^n}_{\text{scale by } |\lambda|^n} \; \underbrace{R(\theta)^n}_{\text{rotation by } n\theta} \vec{x}_0$$

Pure rotation $|\lambda| = 1$ $\qquad \|\vec{x}_n\| = \|\vec{x}_0\| = 1$

Orbits lie on the unit circle $\|\vec{x}\| = 1$



unit circle

$$\vec{x}_n = R(\theta)^n \vec{x}_0 = R(n\theta)\vec{x}_0$$

Note that the motion is periodic only if

$$R(n\theta) = I$$

for some n.

$\vec{x}_0$ is a periodic point of $\vec{x} \mapsto f(\vec{x}) = R\vec{x}$ if

$$\vec{x}_n = \vec{x}_0$$

for some n. Alternately

$$R(n\theta) = I$$

for some n. Suppose

$$\theta = \frac{p}{q}(2\pi)$$

where p, q are integers. $\gamma(\vec{x}_0)$ is periodic with period N if

$$N\theta = 2\pi m$$

for some integer m since then

$$\vec{x}_N = R(N\theta)\vec{x}_0 = R(2\pi m)\vec{x}_0 = \vec{x}_0$$

EX $\theta = \frac{\pi}{4}$ $\frac{p}{q} = \frac{1}{8}$ $\Rightarrow$ $N = 8$ period

EX $\theta = \frac{3\pi}{7}$ $\frac{p}{q} = \frac{3}{14}$ $\Rightarrow$ $N = 14$ period

28?

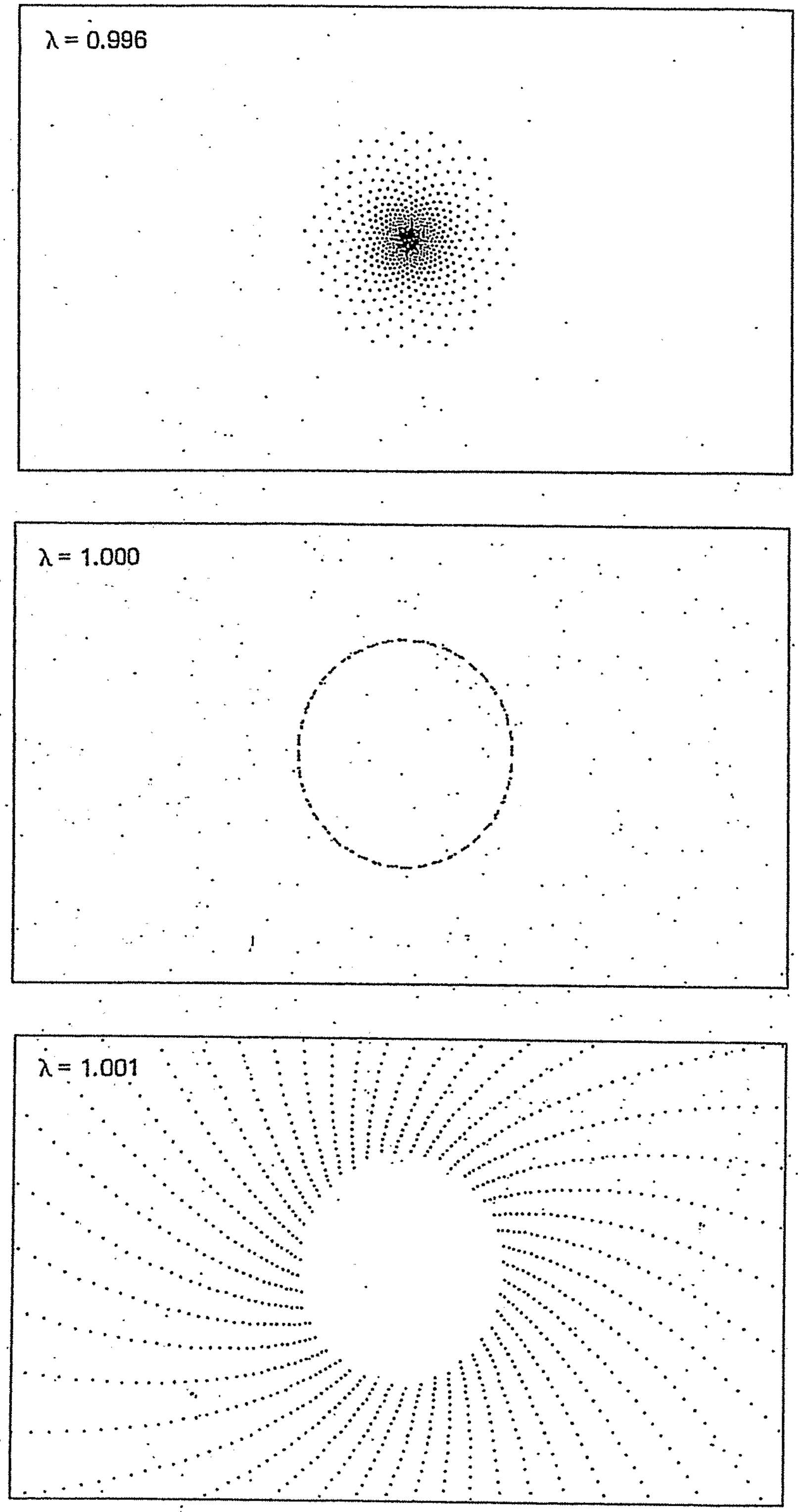


Figure 15.7. *A single (!) orbit of the linear map (15.4) with complex eigenvalues for* $\lambda = 0.996$, $\lambda = 1.000$, *and* $\lambda = 1.001$.

$$A = \lambda R(\theta)$$

Hyperbolicity and Linearization

(1) $$\vec{x}_{n+1} = f(\vec{x}_n)$$

Let $\vec{z}$ be an isolated fixed point of (1)

$$\vec{x}_{n+1} - \vec{z} = f(\vec{x}_n) - \vec{z}$$

(2) $$\vec{x}_{n+1} - \vec{z} = f(\vec{x}_n) - f(\vec{z})$$

Letting $\vec{y}_n = \vec{x}_n - \vec{z}$, eqn (2) becomes

$$\vec{y}_{n+1} = Df(\vec{z})\,\vec{y}_n + O(\|\vec{y}_n\|^2)$$

where upon dropping the higher order terms we get the linearization of (1) about $\vec{z}$

$$\boxed{\vec{y}_{n+1} = Df(\vec{z})\,\vec{y}_n}$$

Here $Df(\vec{z})$ is the Jacobian of f at $\vec{z}$.

Defn A fixed point $\vec{z}$ is hyperbolic if none of the eigenvalues λ of $Df(\vec{z})$ have modulus one. Otherwise, $\vec{z}$ is nonhyperbolic

EXAMPLE Delayed Logistic map

$$f(\vec{x}) = \begin{pmatrix} y \\ \mu y(1-x) \end{pmatrix} \qquad \mu > 0$$

has two fixed points

$$\vec{z}_0 = (0,0) \qquad \vec{z}_1 = \left(1-\frac{1}{\mu}, 1-\frac{1}{\mu}\right)$$

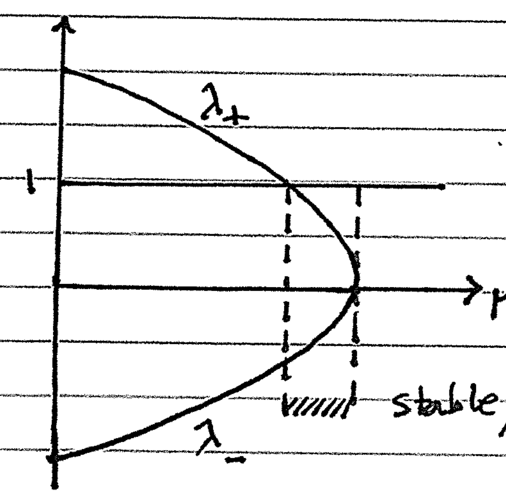
Linear analysis at $\vec{z}_0$

$$Df(\vec{z}_0) = \begin{bmatrix} 0 & 1 \\ 0 & \mu \end{bmatrix} \qquad \lambda_1 = 0 \quad \lambda_2 = \mu$$

hyperbolic as long as $\mu \neq \pm 1$. Stable if $|\mu| < 1$.

Linear analysis at $\vec{z}_1$

$$Df(\vec{z}_1) = \begin{bmatrix} 0 & 1 \\ 1-\frac{1}{\mu} & 1 \end{bmatrix} \qquad \lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{5-4\mu}$$



stable, $|\lambda_k| < 1$. $\forall k$

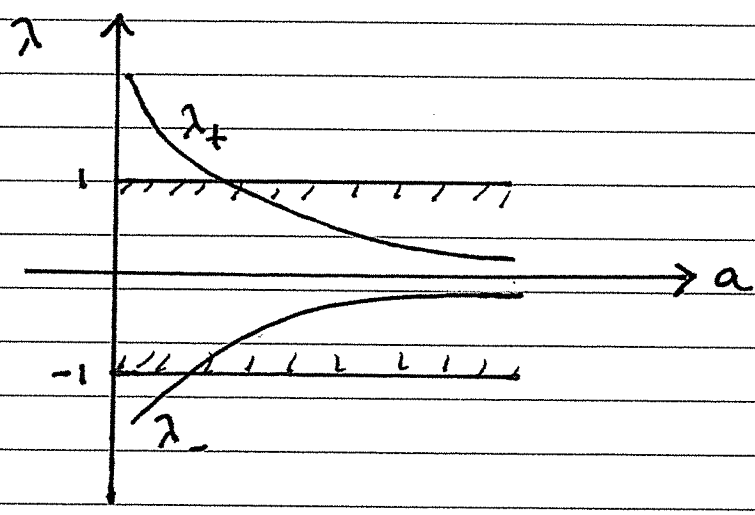
EXAMPLE Henon map

$$f(\vec{x}) = \begin{pmatrix} y + 1 - ax^2 \\ bx \end{pmatrix}$$

Fixed point $\vec{z} = (x, y)^T$ hay $y = bx$ where x is a root of

$$ax^2 + (1-b)x - 1 = 0$$

The eigenvalues of $Df(\vec{z})$ are $\lambda_\pm(a,b) \in \mathbb{R}$



$$W^s(\vec{z}) \neq \mathbb{R}^2$$

Henon map example

For the specific map

$$f(\vec{x}) = (2.12 - x^2 - 0.3y, x)$$

$\det Df(\vec{x}) = 0.3$ hence f is contracting. For fixed point $\vec{z} \equiv (0.944, 0.944)$ the eigenvalues of $Df(\vec{z})$ are

$$\lambda_1 = -1.17 \qquad \lambda_2 = -0.175$$

(unstable) (stable)

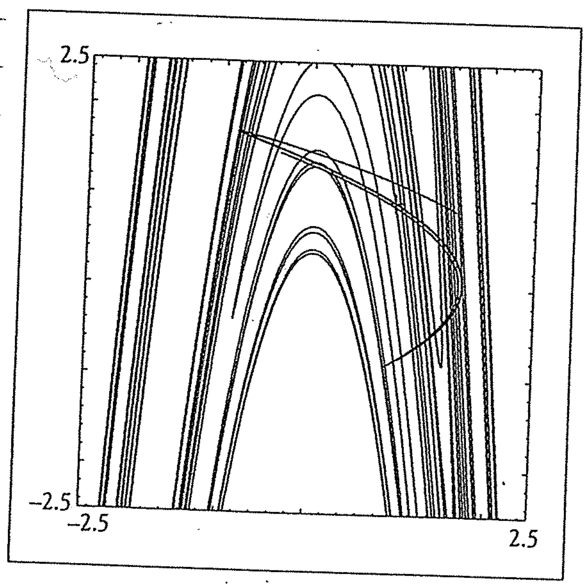
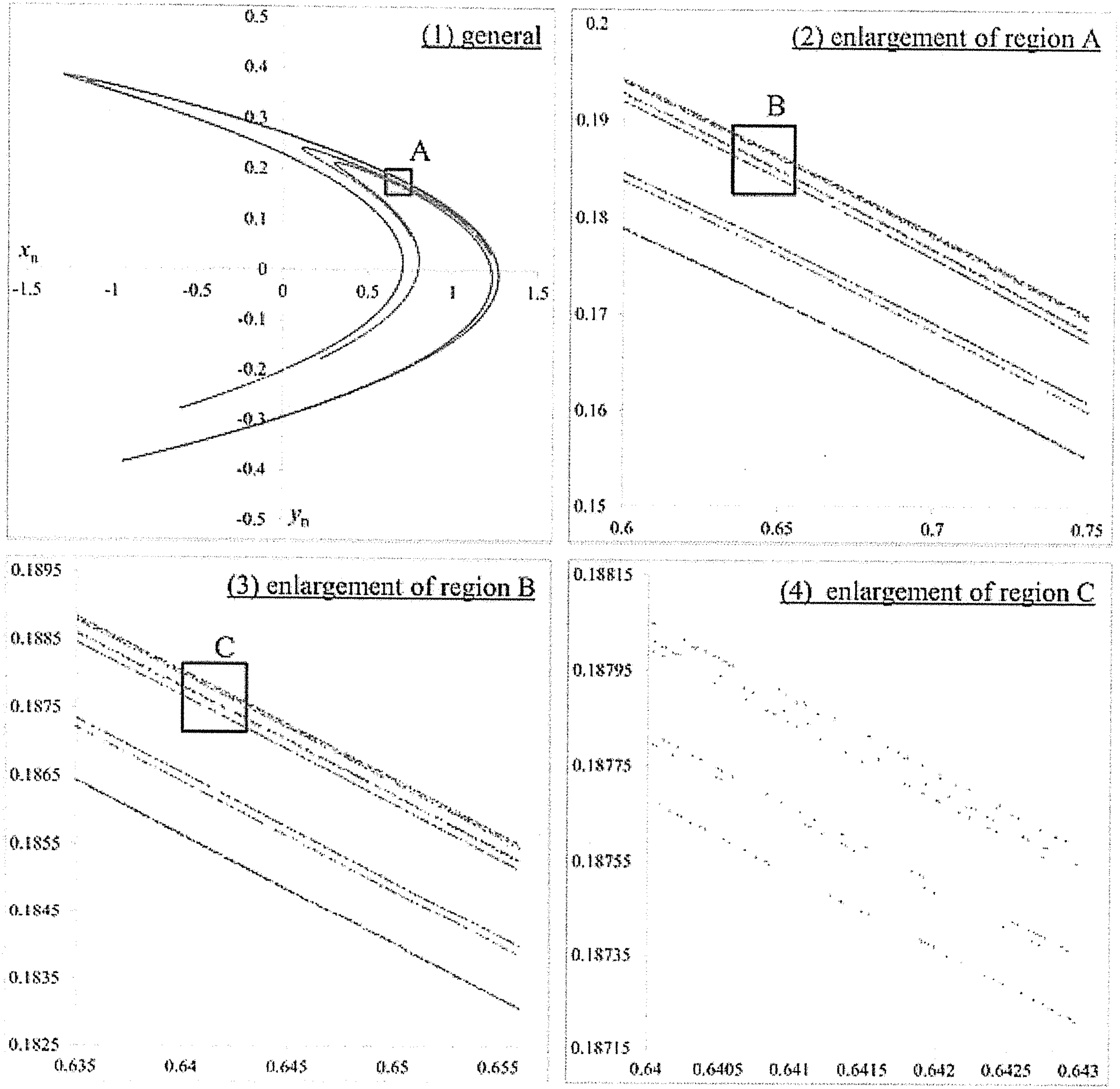


Figure 10.7 Stable and unstable manifolds for a fixed point saddle of the Hénon map $f(x, y) = (2.12 - x^2 - .3y, x)$.
The fixed point is marked with a cross. The unstable manifold is S-shaped; the stable manifold is primarily vertical.


(1) general
A
x_n
y_n
(2) enlargement of region A
B
(3) enlargement of region B
C
(4) enlargement of region C

Poincaré-Andronou-Hopf Bifurcation

Canonical example $\quad \mathbb{X} = (x, y) \quad , \quad r = \sqrt{x^2 + y^2}$

$$(1) \qquad F(\mathbb{X}) = (\lambda - r^2)\, R(\omega)\, \mathbb{X}$$

where $R(\omega)$ is rotation by an angle ω (counterclockwise)
In polar coordinates $\mathbb{X} = (r, \theta)$ so that $\mathbb{X} \mapsto F(\mathbb{X})$ has the polar representation:

$$(2) \qquad \begin{pmatrix} r \\ \theta \end{pmatrix} \xmapsto[F]{} \begin{pmatrix} \lambda r - r^3 \\ \theta + \omega \end{pmatrix}$$

Notice that the dynamics in (2) show that the circle

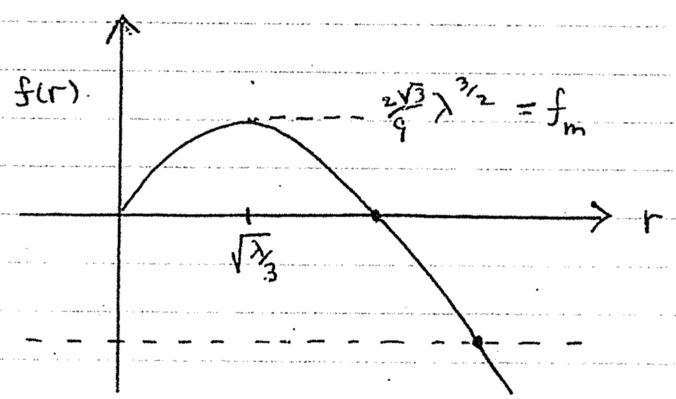
$$C = \{ (r, \theta) : \ r = \sqrt{\lambda - 1} \} \qquad \lambda > 1.$$

is invariant under F, i.e

$$r = \lambda r - r^3 = f(r) \quad \text{if} \quad r = \sqrt{\lambda - 1}$$

so that $\quad F(C) = C.$

Now note that $\quad \| F(\mathbb{X}) \| = | r(\lambda - r^2) | \equiv | f(r) |$



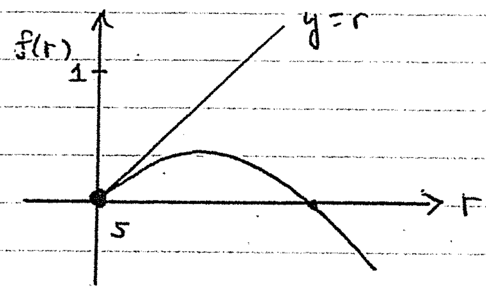
$\lambda > 0$

It is easily verified that $\forall \lambda \in (0, \lambda_0)$, $\lambda_0 = \frac{3}{4} 4^{2/3} \simeq 1.9$

$$1 > f_m > 0 \quad , \quad 0 < \lambda < \lambda_0$$

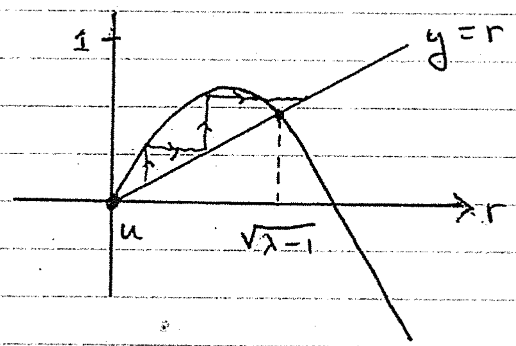
So that for $\lambda \in (0, \lambda_0)$,

(a)



$0 < \lambda < 1$

(b)



$1 < \lambda < \lambda_0$

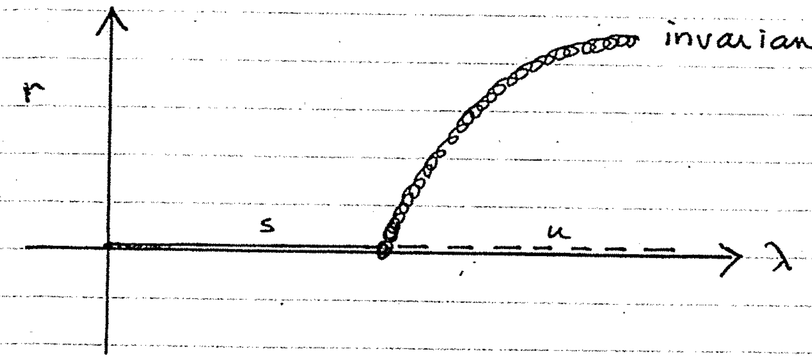
Consider $r \longmapsto f(r)$ as a map with fixed point $\bar{r}(\lambda) = \sqrt{\lambda - 1}$

$$f'(r) = \lambda - 3r^2$$

$$f'(\bar{r}) = 3 - 2\lambda$$

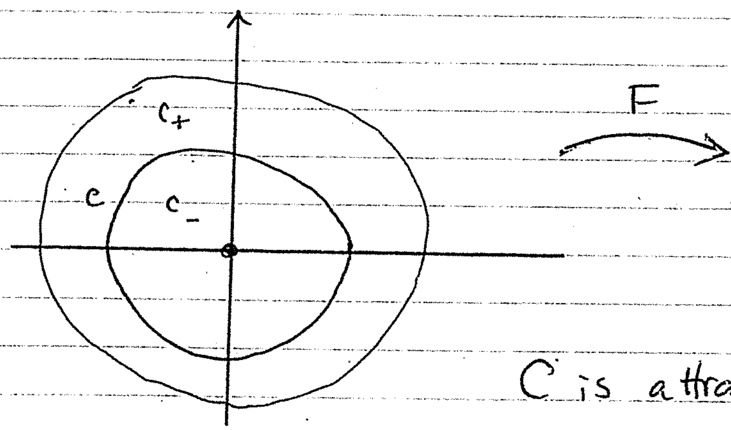
$$|f'(\bar{r})| < 1 \qquad \forall \lambda \in (1, 2)$$

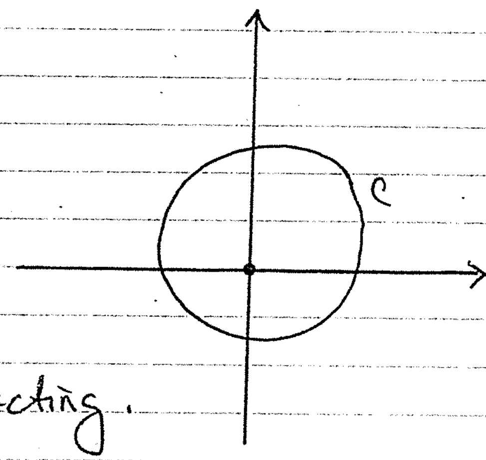
From which we deduce a bifurcation diagram for F!



For $\lambda \in (1,2)$ we have for F,

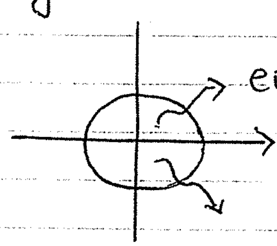
$$
\begin{aligned}
C_0 &= \{(r,\theta): \ r = 0\} \\
C_- &= \{(r,\theta): \ r < \sqrt{\lambda - 1}\} \\
C &= \{(r,\theta): \ r = \sqrt{\lambda - 1}\} \\
C_+ &= \{(r,\theta): \sqrt{\lambda-1} < r < r_0(\lambda)\}
\end{aligned}
$$





C is attracting.

<u>Remark</u> If $\lambda(\mu)$ is the eigenvalue of Jac of $\underline{x} \mapsto F(\underline{x},\mu)$ (for appropriate F) and $|\lambda(\mu^*)| = 1$ at $\mu = \mu^*$, eigvals of $DF(0,\mu)$



evals cross transverse

then this sort of bif. occurs.