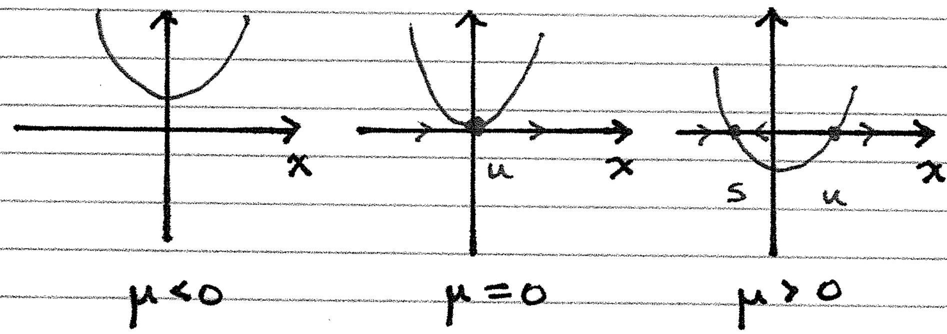
Saddle Node (SN) Bifurcations – introduction

$$\dot{x} = f(x,\mu) = x^2 - \mu \qquad x, \mu \in \mathbb{R}$$

is the simplest system with a saddle node bifurcation. Below are graphs of $f(x,\mu)$

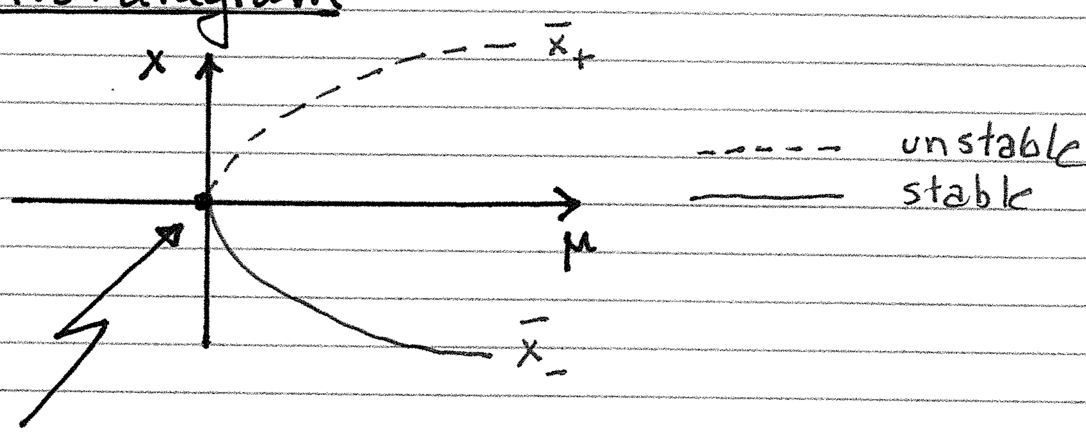


Here one can solve $f(x,\mu) = 0$ for x to find two branches of fixed points

$$\bar{x}_+ = \sqrt{\mu} \qquad \text{unstable}$$

$$\bar{x}_- = -\sqrt{\mu} \qquad \text{stable}$$

This collective information yields a bifurcation diagram



$(x^*, \mu^*) = (0,0)$ is SN bifurcation point

EXAMPLE Sometimes you can't solve $f(x,\mu)=0$ for x

$$\dot{x} = f(x,\mu) = \mu - x - e^{-x}$$

Instead of solving $f(x,\mu)=0$ for x we find the locus of fixed points by solving $f(x,\mu)=0$ for μ

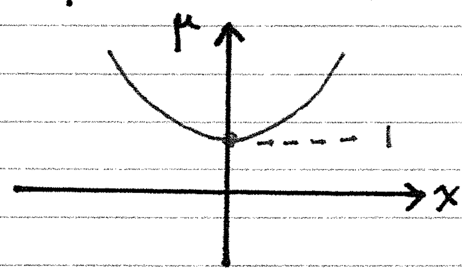
$$\bar{\mu}(x) = x + e^{-x}$$

Here we use calculus to graph $\bar{\mu}(x)$

$$\bar{\mu}'(x) = 1 - e^{-x} \qquad \bar{\mu}'(0) = 0$$

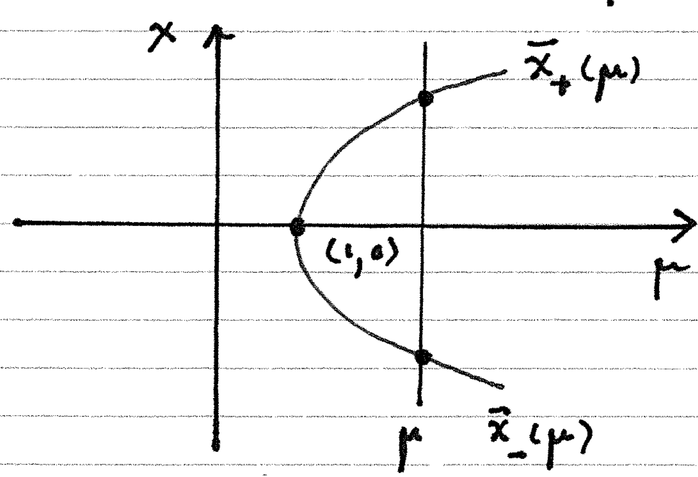
$$\bar{\mu}''(x) = e^{-x} > 0 \qquad \text{concave up. } \cup$$

Noting $\bar{\mu}(0) = 1$ we have the following qualitatively accurate graph



locus of fixed points in (x,μ)-plane

Reflect this about $\mu = x$ line to get fixed point (branches) as a function of μ



Fixed point location only. No stability

To get the stability we need to plot $f(x,\mu)$ for various μ, (see prev. figure). These plots are phase portaits.

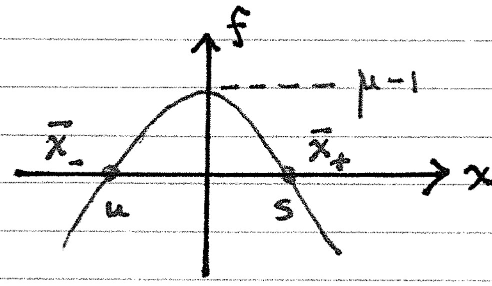
$$f(x,\mu) = \mu - x - e^{-x} = \mu - \bar{\mu}(x)$$

Noting

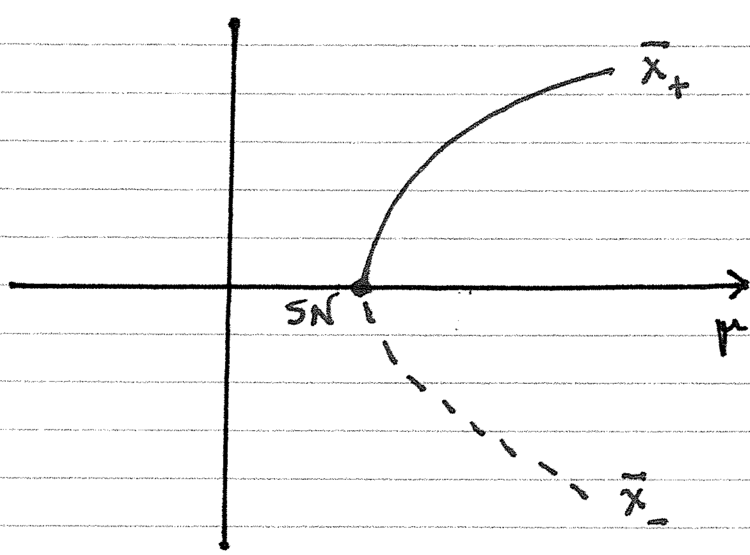
$$f(0,\mu) = \mu - 1$$

$$\lim_{x \to \pm\infty} f(x,\mu) = -\infty \qquad (\mu > 1)$$

For any $\mu > 1$



Collectively we have the following bif. diag. for $\dot{x} = f(x,\mu)$



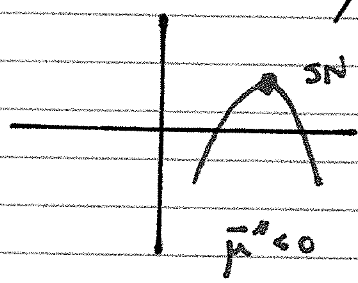
(1,0) is a SN bif. point

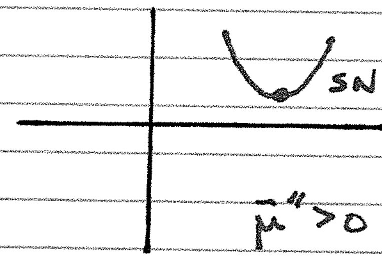
Saddle Nodes with Quadratic Tangency (Definition)

$$\boxed{\begin{aligned} f(x, \bar{\mu}(x)) &= 0 \\ \bar{\mu}'(x^*) &= 0 \\ \bar{\mu}''(x^*) &\neq 0 \end{aligned}}$$

(x^*, μ^*) SN-bif.

When satisfied the SN bif is said to have quadratic tangency





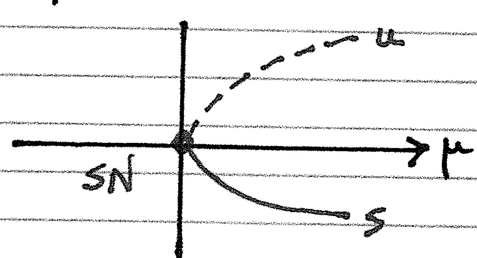
EXAMPLE $\dot{x} = x^2 + \mu$ has SN at (x^*, μ^*) and has quadratic tangency

EXAMPLE $\dot{x} = x^4 - \mu$ has SN but not quadratic tang.

$\bar{\mu}(x) = x^4$

$\bar{\mu}'(x) = 4x^3$ $\qquad$ $\bar{\mu}'(x) = 0$ ✓

$\bar{\mu}''(x) = 12x^2$ $\qquad$ $\bar{\mu}''(0) = 0$ ✗

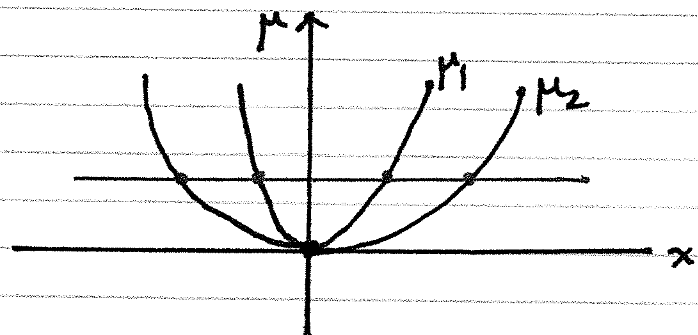


Atypical multibanch saddle node

$$\dot{x} = f(x,\mu) = (x^2-\mu)(x^2-4\mu)$$

Note that $\mu < 0 \Rightarrow f(x,\mu) > 0$ so that there are no fixed points for negative μ.
When $\mu > 0$ the locus of fixed points are

$$\mu = \mu_1(x) = x^2 \qquad \mu = \mu_2(x) = \frac{1}{4}x^2$$

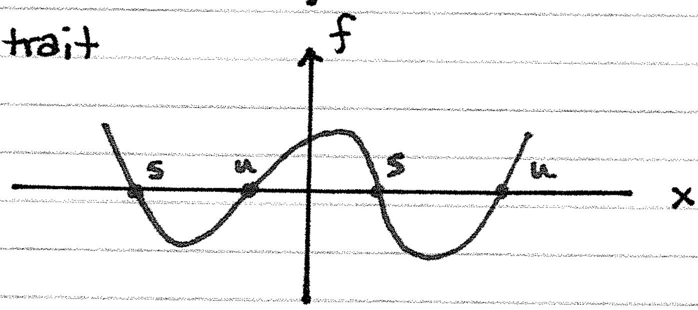


Fixed Points

$\bar{x} = \pm\sqrt{\mu}$

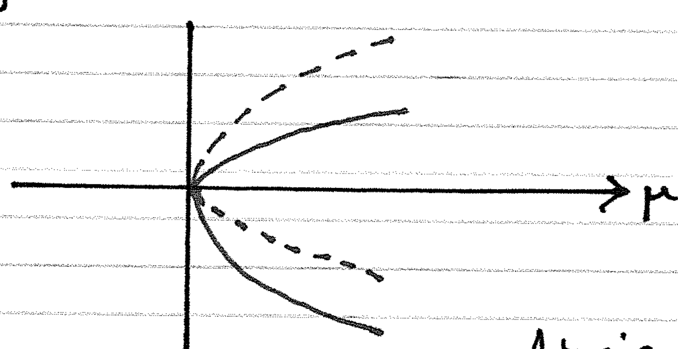
$\bar{x} = \pm 2\sqrt{\mu}$

Phase Portrait for $\mu > 0$



Combining the information from both figures

Bifurcation Diagram.



Atypical SN

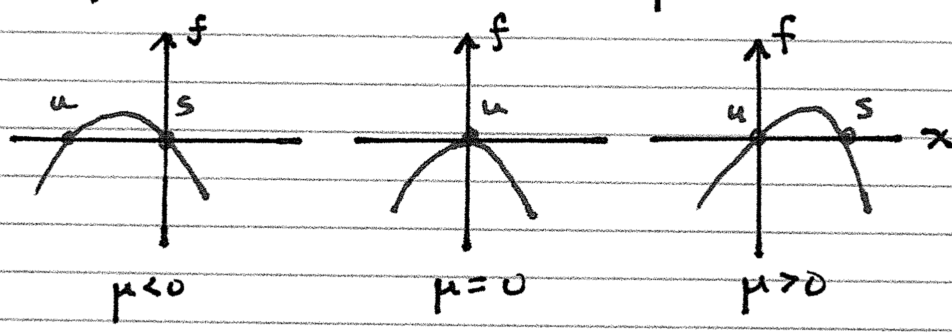
Transcritical Bifurcation (exchange of stability)

$$\dot{x} = f(x,\mu) = \mu x - x^2$$

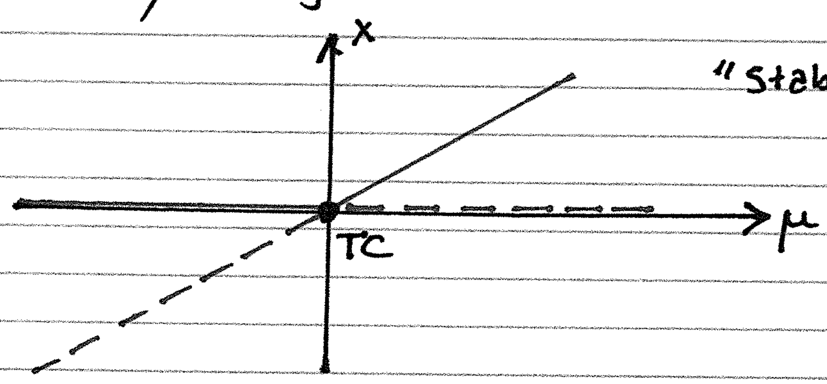
Has two branches of fixed points

$$\bar{x}_1 = 0 \qquad \bar{x}_2 = \mu$$

Phase portraits for various μ values



Collectively we get the bifurcation diagram



Transcritical Bifurcation at $(x^*, \mu^*) = (0,0)$

EXAMPLE If concentrations of A and B are nearly fixed and

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftarrows}} 2X \qquad X + B \xrightarrow{k_2} C$$

then the concentration x of X is given by

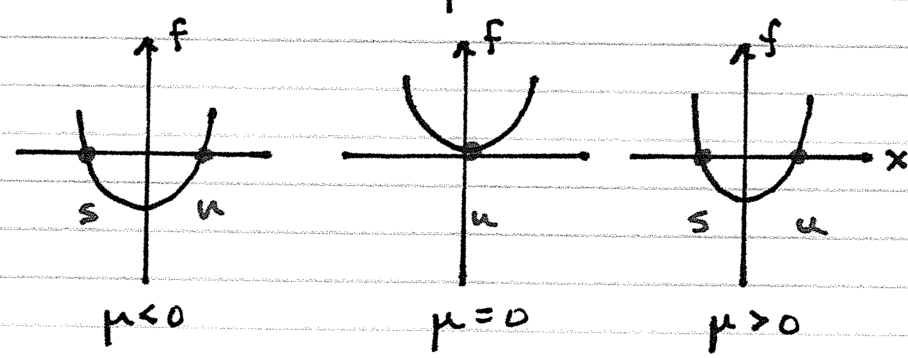
$$\dot{x} = c_1 x - c_2 x^2$$

which has a TC-bif at $x=0$

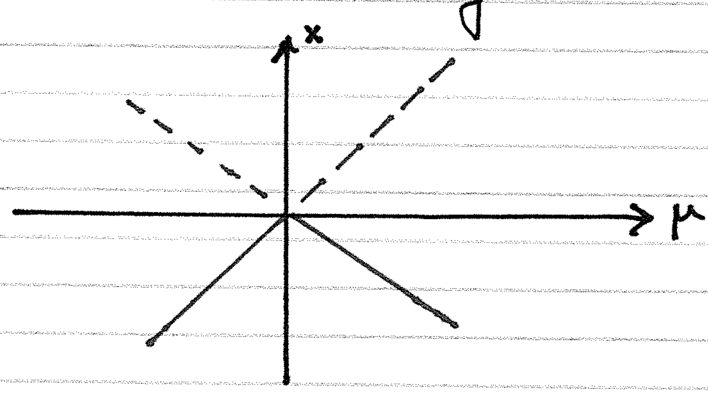
EXAMPLE Another simple example

$$\dot{x} = x^2 - \mu^2$$

has branches $x = \pm\mu$.



yields the bifurcation diagram



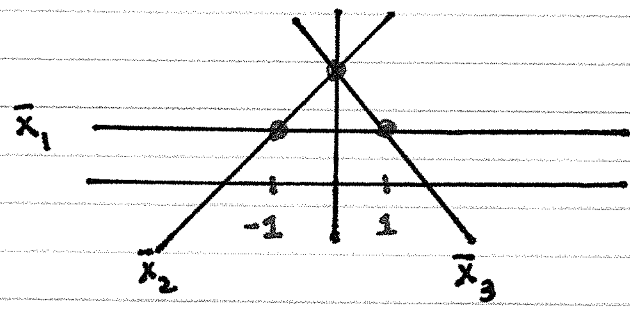
EXAMPLE Multiple Transcritical Bifurcations

$$\dot{x} = (x-1)((x-2)^2-\mu^2) = f(x,\mu)$$

has three branches of equilibria

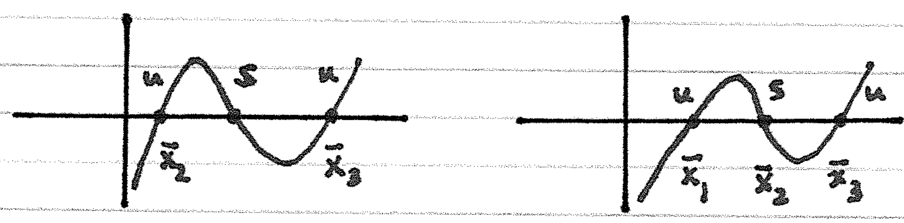
$$\bar{x}_1 = 1 \qquad \bar{x}_2 = 2+\mu \qquad \bar{x}_3 = 2-\mu$$

Locus of fixed points in (μ, x)-plane



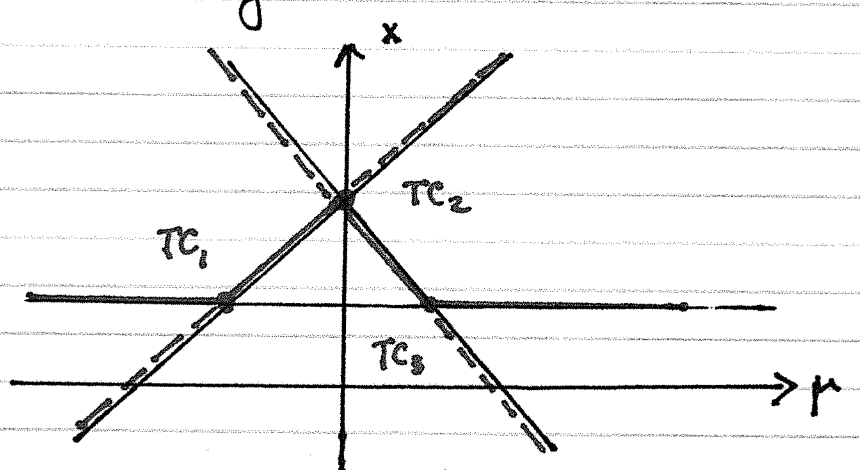
• TC bifurcation points

For stability must plot $f(x,\mu)$ at various μ

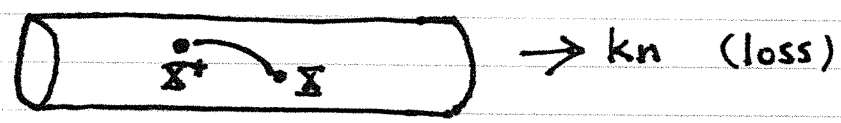


plotted with software.

Bifurcation diagram



Transcritical Bifurcations in lasers



Excited atoms X^+ give off photons when they go to a lower energy state X.

$$n(t) = \#\text{ photons in laserfield}$$

$$N(t) = \#\text{ excited atoms}$$

At rest, pump keeps $N(t)$ at N_0, and assume

(1) $$N = N_0 - \alpha n$$ reduces due to photons lost when $X^+ \to X$

Then

$$\dot{n} = \text{gain} - \text{loss}$$

(2) $$\dot{n} = GnN - kn$$

Given (1) and G = gain coeff we have

$$\boxed{\dot{n} = (GN_0 - k)n - (\alpha G)n^2}$$

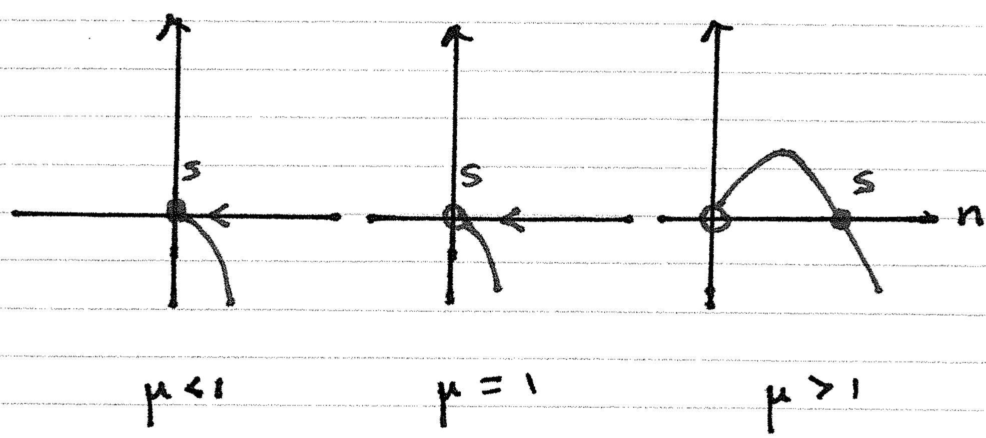
Here we have a quadratic $f(n)$ of the form

$$f(n) = \alpha n - \beta n^2$$

Plots of f(n) for various parameters

$$\mu \equiv \frac{N_0 G}{k}$$

Then



Results in