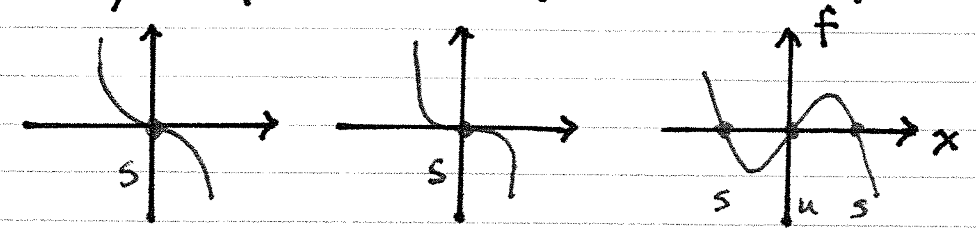
Pitchfork Bifurcations - introduction

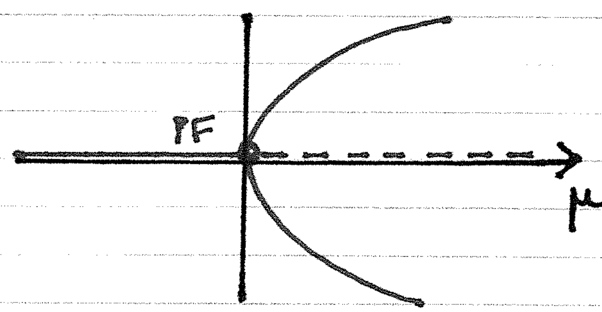
$$\dot{x} = f(x,\mu) = x(\mu - x^2) \qquad \text{cubic}$$

$\bar{x} = 0$ is always a fixed point. Has an additional two only if parameter $\mu > 0$. Phase portraits



$\mu < 0$ $\qquad$ $\mu = 0$ $\qquad$ $\mu > 0$

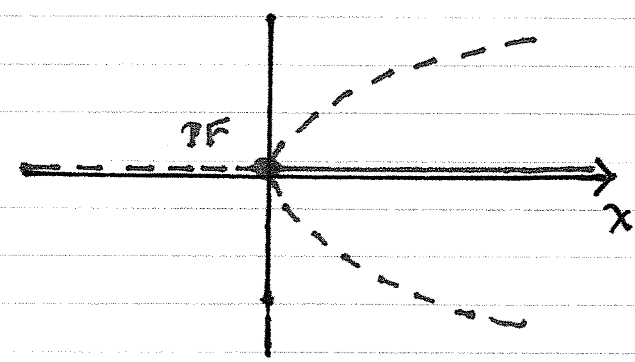
results the bifurcation diagram:



Supercritical Pitchfork bifurcation

(Quadratic tangency)

Were the problem altered to $\dot{x} = -\mu x + x^3$



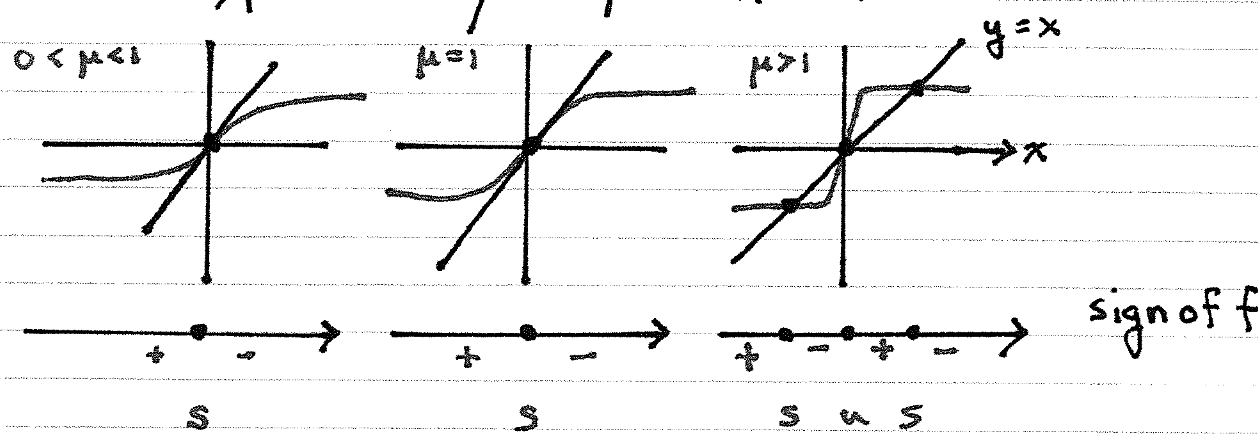
Subcritical PF bif. at $(x^*, \mu^*) = (0,0)$

EXAMPLE (Neural net models)

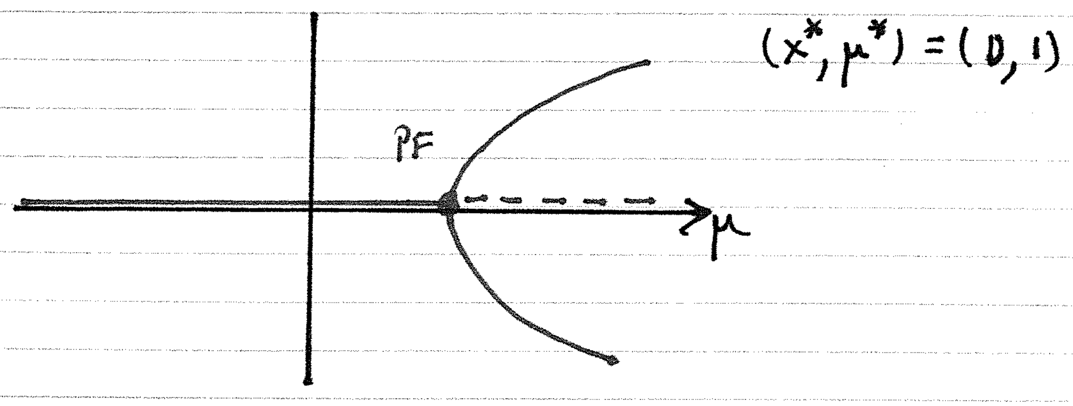
$$\dot{x} = -x + \mu \tanh x = f(x,\mu)$$

Find how fixed pts (roots of f) vary with μ and how sign of f changes.

Note $f(x,\mu) > 0$ only if $\mu \tanh x > x$



Hence the system has a supercritical Pitch Fork bifurcation

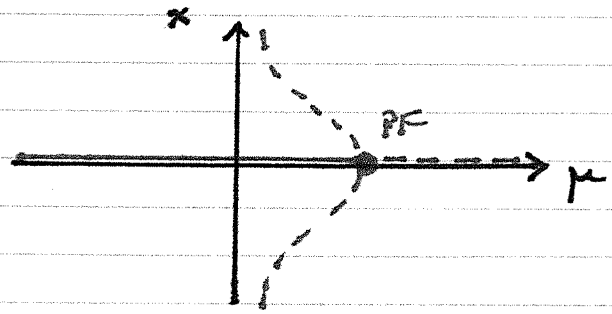


Note the locus of equilibria can be found by solving $f(x,\mu) = 0$ for μ

$$\mu = \frac{x}{\tanh x} \qquad x \neq 0$$

EXAMPLES Pitchforks with quadratic tangencies

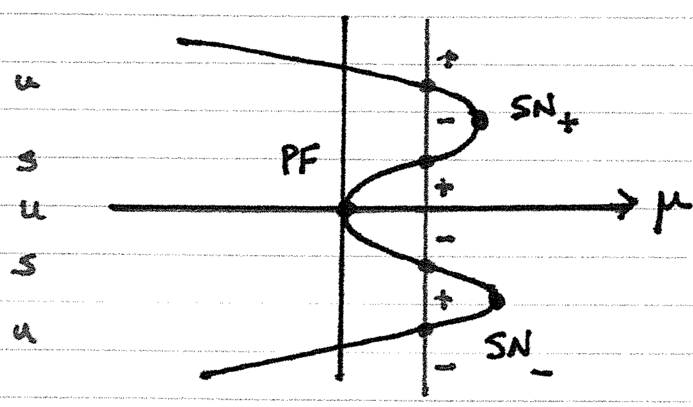
(a) $f(x,\mu) = x(\mu - e^{-x^2})$



Subcritical PF at

$(\mu^*, x^*) = (1, 0)$

(b) $f(x,\mu) = x(\mu - \sin x^2)$



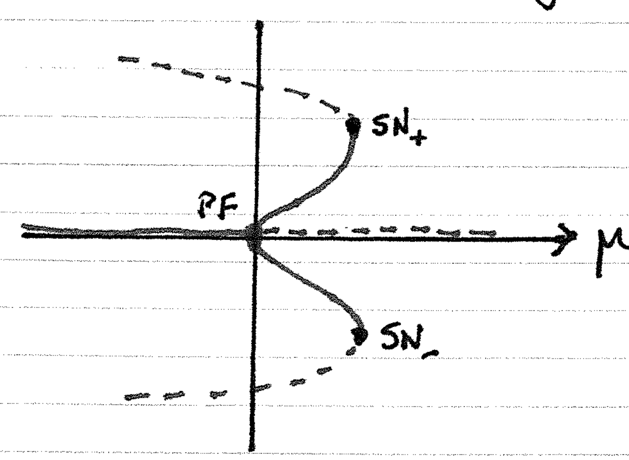
For the indicated μ constant line the sign of

$f = x(\mu - \sin x^2)$

can change only at

$x = 0 \qquad \mu = \sin x^2$

From this we deduce the signs of $f(x,\mu)$ at various x along μ constant. Can therefore conclude the following bifurcation diagram



Supercritical PF

Two SN (for the range of x shown)

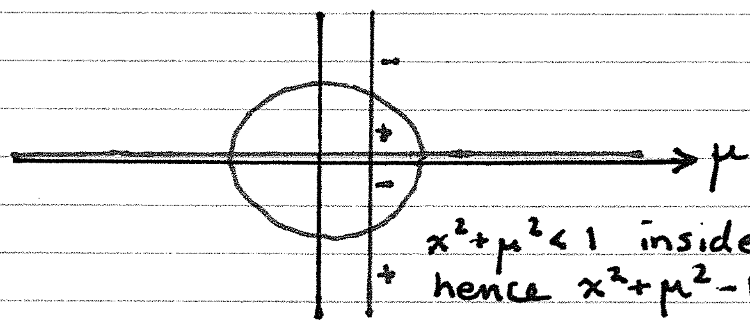
EXAMPLE Multiple Pitchforks

$$\dot{x} = -x\,(x^2 + \mu^2 - 1)$$

has two "branches" of fixed points

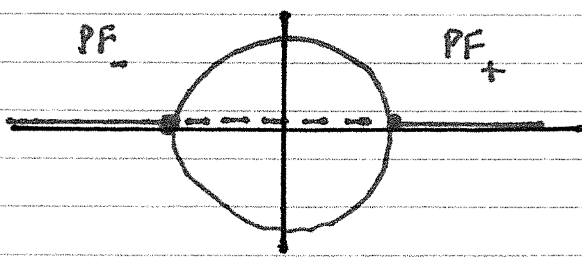
$$\bar{x} = 0 \qquad \bar{x}^2 + \bar{\mu}^2 = 1$$

The latter is a circle of radius 1. The locus of fixed points is



$x^2 + \mu^2 < 1$ inside circle
hence $x^2 + \mu^2 - 1 < 0$ there

Can deduce signs of $f(x, \mu)$ for various μ and hence fixed point stability:



Structural Stability

Our three generic bifurcations are

$$\dot{x} = x^2 - \mu \qquad \text{SN}$$

$$\dot{x} = \mu x - x^2 \qquad \text{TC}$$

$$\dot{x} = x(\mu - x^2) \qquad \text{PF}$$

We give a casual explanation of the notion of structural stability. Basically a system is structurally stable if the bifurcation diagrams of

$$\dot{x} = f(x,\mu)$$

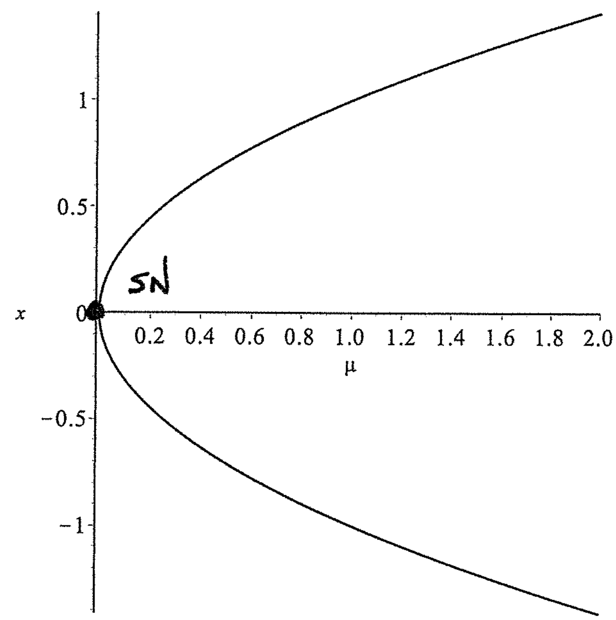
$$\dot{x} = f(x,\mu) + \underbrace{\varepsilon g(x,\mu)}_{\text{small } \varepsilon \ll 1 \text{ perturbation}}$$

are "qualitatively" the same for ε small enough and g "well behaved".

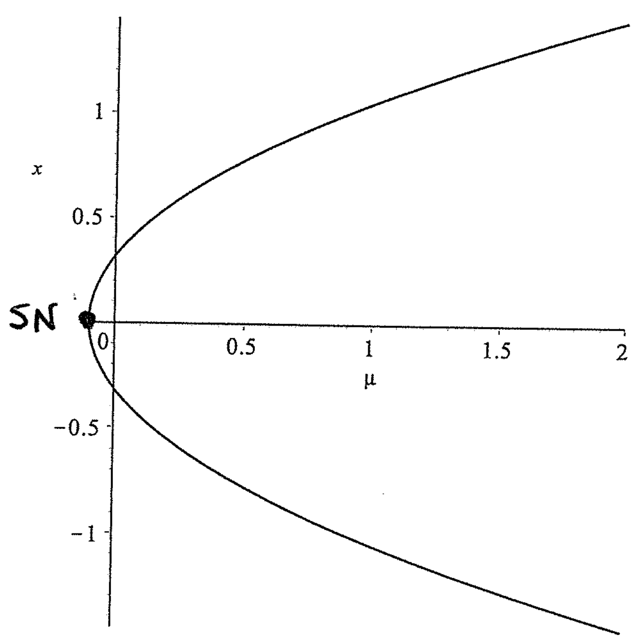
Turns out (next three pages) that only the SN is structurally stable.

The notion means that if you "jiggle" the system, the dynamics remain intact

Structurally Stable SN $f(x, \mu) = \mu - x^2$



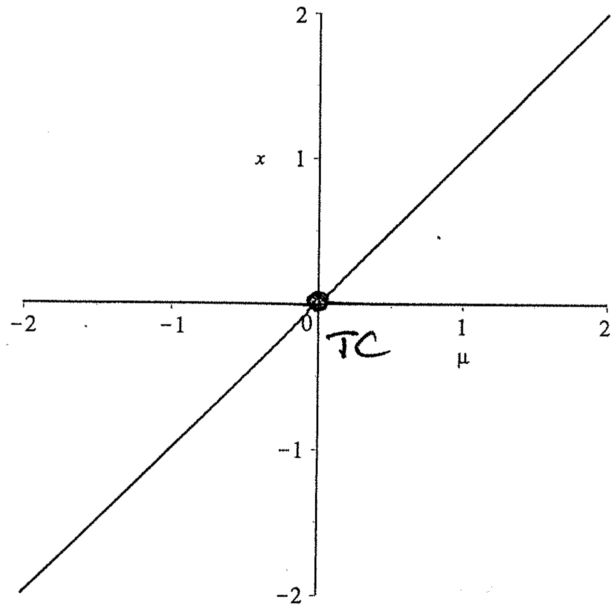
$\dot{x} = f(x, \mu)$



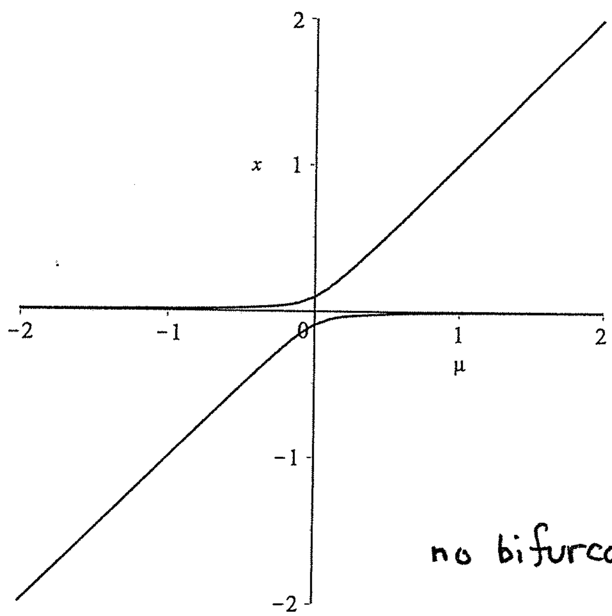
$\dot{x} = f(x, \mu) + \varepsilon$

$\varepsilon = 0.$

Structurally Unstable TC : $f(x,\mu) = \mu x - x^2$



$\dot{x} = f(x,\mu)$

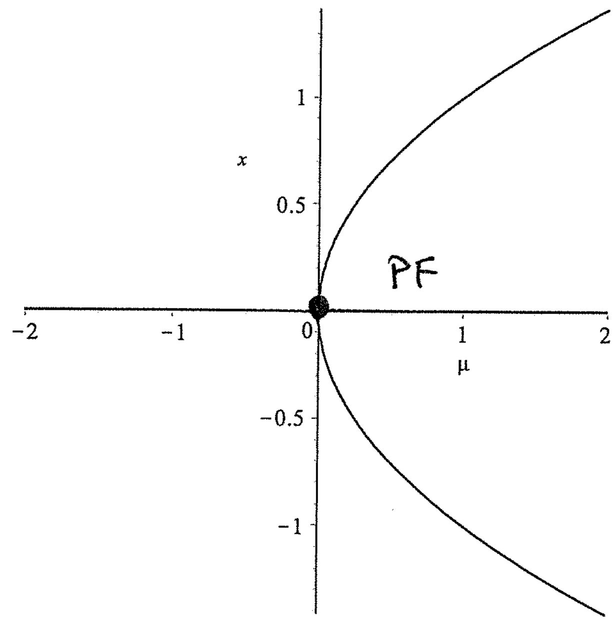


$\dot{x} = f(x,\mu) + \varepsilon$

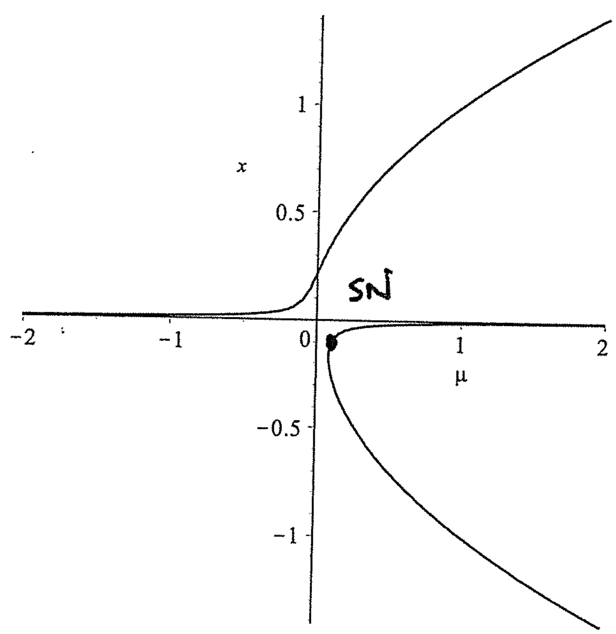
$\varepsilon = 0.01$

no bifurcations!

Structurally Unstable PF: $f(x,\mu) = x(\mu - x^2)$



$\dot{x} = f(x,\mu)$



$\dot{x} = f(x,\mu) + \varepsilon$

$\varepsilon = 0.01$