Pitchfork Bifurcations - introduction

\[ \dot{x} = f(x, \mu) = x(\mu - x^2) \quad \text{cubic} \]

\[ \dot{x} = 0 \] is always a fixed point. Has an additional two only if parameter \( \mu > 0 \). Phase portraits

\[ \mu < 0 \quad \mu = 0 \quad \mu > 0 \]

results the bifurcation diagram:

\[ \text{Supercritical} \]
\[ \text{Pitchfork bifurcation} \]
\[ \text{(quadratic tangency)} \]

Were the problem altered to \[ \dot{x} = -\mu x + x^3 \]

\[ \text{Subcritical PF} \]
\[ \text{bif. at } (x^*, \mu^*) = (0, 0) \]
**Example**  (Neural net models)

\[ \dot{x} = -x + \mu \tanh x = f(x, \mu) \]

Find how fixed pts (roots of \( f \)) vary with \( \mu \) and how sign of \( f \) changes.

Note \( f(x, \mu) > 0 \) only if \( \mu \tanh x > x \)

\[ \begin{align*}
0 < \mu < 1 & \\
\mu = 1 & \\
\mu > 1 & \\
\end{align*} \]

Hence the system has a supercritical Pitch Fork bifurcation

\[ (x^*, \mu^*) = (0, 1) \]

Note the locus of equilibria can be found by solving \( f(x, \mu) = 0 \) for \( \mu \)

\[ \mu = \frac{x}{\tanh x}, \quad x \neq 0 \]
EXAMPLES  Pitchforks with quadratic tangencies

(a) \( f(x, \mu) = x(\mu - e^{-x^2}) \)

Subcritical PF at 
\( (\mu^*, x^*) = (1, 0) \)

(b) \( f(x, \mu) = x(\mu - \sin x^2) \)

For the indicated \( \mu \) constant line the sign of \( f = x(\mu - \sin x^2) \) can change only at 
\( x = 0 \quad \mu = \sin x^2 \)

From this we deduce the signs of \( f(x, \mu) \) at various \( x \) along \( \mu \) constant. Can therefore conclude the following bifurcation diagram

Supercritical PF

Two SN (for the range of \( x \) shown)
EXAMPLE  Multiple Pitchforks

\[ \dot{x} = -x \left( x^2 + \mu^2 - 1 \right) \]

has two "branches" of fixed points

\[ \overline{x} = 0 \quad \overline{x}^2 + \overline{\mu}^2 = 1 \]

The latter is a circle of radius 1. The locus of fixed points is

\[ x^2 + \mu^2 < 1 \text{ inside circle} \]

hence \( x^2 + \mu^2 - 1 < 0 \) there

Can deduce signs of \( f(x, \mu) \) for various \( \mu \)
and hence fixed point stability:
Structural Stability

Our three generic bifurcations are

\[ \dot{x} = x^2 - \mu \quad \text{SN} \]
\[ \dot{x} = \mu x - x^2 \quad \text{TC} \]
\[ \dot{x} = x(\mu - x^2) \quad \text{PF} \]

We give a casual explanation of the notion of structural stability. Basically a system is structurally stable if the bifurcation diagrams of

\[ \dot{x} = f(x, \mu) \]
\[ \dot{x} = f(x, \mu) + \varepsilon g(x, \mu) \]

small \( \varepsilon \ll 1 \) perturbation

are "qualitatively" the same for \( \varepsilon \) small enough and \( g \) "well behaved".

Turns out (next three pages) that only the SN is structurally stable.

The notion means that if you "jiggle" the system, the dynamics remain intact.
Structurally Stable SN $f(x, \mu) = \mu - x^2$

$x = f(x, \mu)$

$\dot{x} = f(x, \mu) + \epsilon$

$\epsilon = 0$
Structurally Unstable IC: \( f(x, \mu) = \mu x - x^2 \)

\[
\dot{x} = f(x, \mu)
\]

\[
\dot{x} = f(x, \mu) + \varepsilon
\]

\( \varepsilon = 0.01 \)

no bifurcations!
Structurally Unstable PF: \( f(x, \mu) = x(\mu - x^2) \)

\[ x = f(x, \mu) \]

\[ x = f(x, \mu) + \varepsilon \]

\( \varepsilon = 0.01 \)