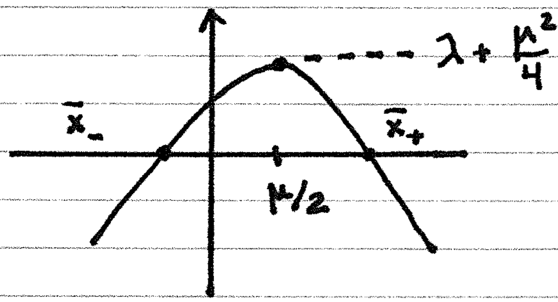
Perturbed TC bifurcations

$$\boxed{\dot{x} = f(x, \mu, \lambda) \equiv \lambda + \mu x - x^2}$$

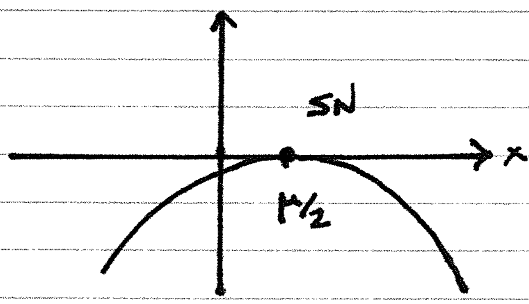
Here f depends on two parameters (μ, λ) and the system has a TC when $\lambda = 0$.

2 roots



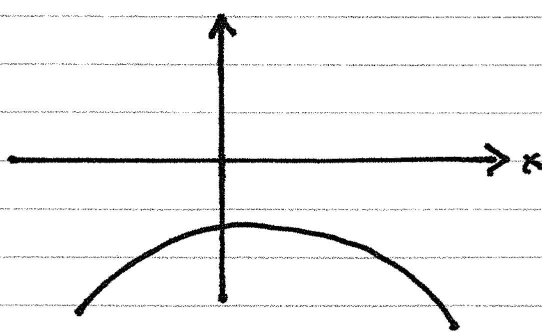
$$\bar{x}_\pm = \tfrac{1}{2}\mu \pm \tfrac{1}{2}\sqrt{\mu^2 + 4\lambda}$$

1 root



$$\lambda = -\tfrac{1}{4}\mu^2$$

no root



$$\lambda < -\tfrac{1}{4}\mu^2$$

Locating nonhyperbolic (bifurcation) point

$$(1) \qquad f(x, \mu, \lambda) = 0$$

$$(2) \qquad f_x(x, \mu, \lambda) = 0$$

are two eqns for three unknowns. Explicitly

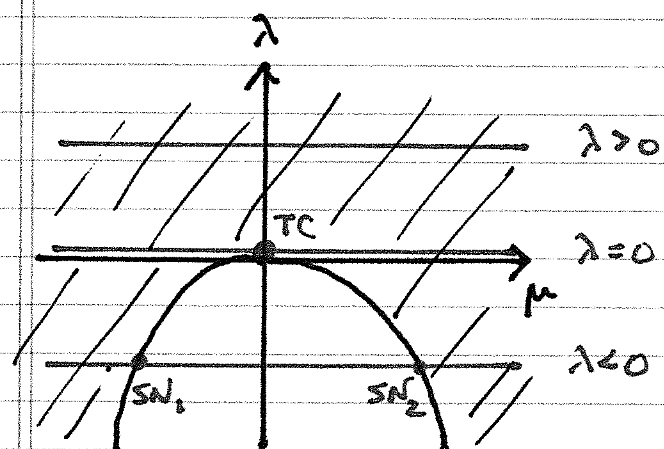
$$(3) \qquad \lambda + \mu x - x^2 = 0$$

$$(4) \qquad \mu - 2x = 0$$

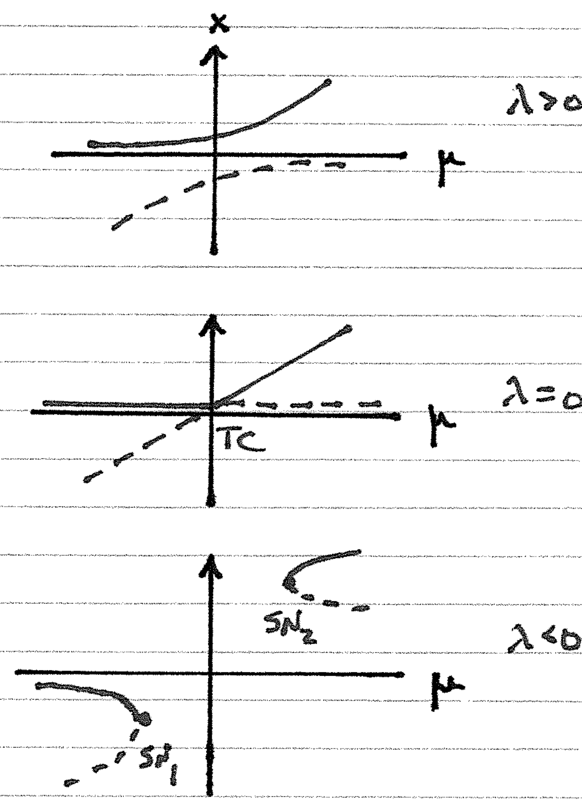
Eliminate x to get the locus of nonhyp. pts.

$$\boxed{\lambda = -\frac{1}{4}\mu^2}$$

Bifurcation diagrams (without work)



$\lambda = -\frac{1}{4}\mu^2$



Detail for fixed $\lambda < 0$ using normal forms

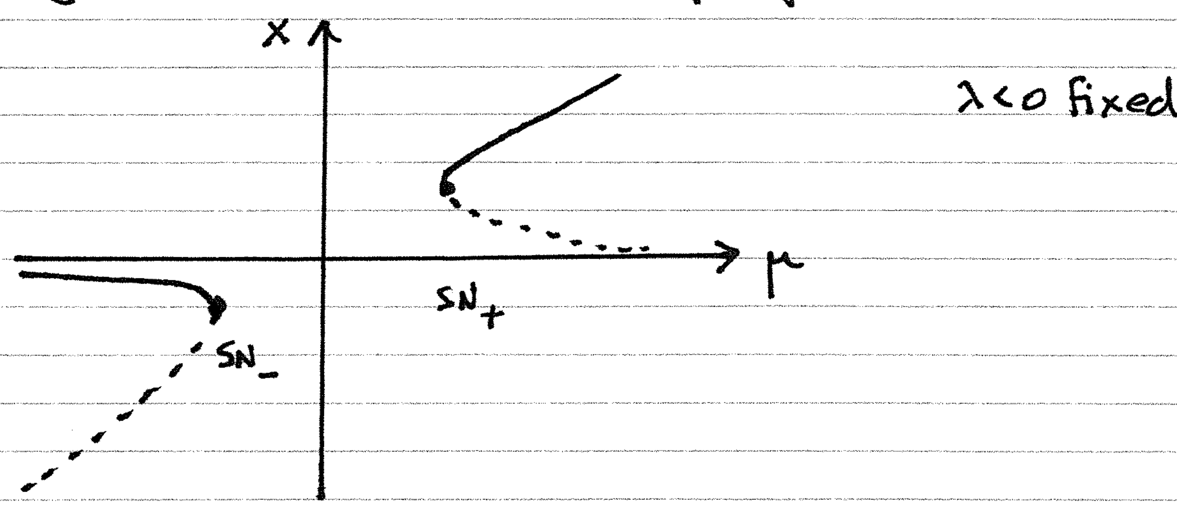
From the theorem one requires the following for a saddle node.

$$\left|\begin{array}{lcll} f & = & \lambda + \mu x - x^2 & = 0 \\ f_x & = & \mu - 2x & = 0 \\ f_\mu & = & x & \neq 0 \\ f_{xx} & = & -2 & \neq 0 \end{array}\right| \quad \text{SN conditions}$$

For $\lambda < 0$ the first two eqns imply

$$\mu = -2\sqrt{|\lambda|} \qquad \mu = 2\sqrt{|\lambda|}$$

Plotting $f = 0$ (and considering signs of f)



$$\boxed{\bar{\mu} = \frac{x^2 - \lambda}{x}}$$ fixed point location.

Perturbed Pitchfork

$$\boxed{\dot{x} = f(x, \mu, \lambda) = \lambda + \mu x - x^3}$$

When $\lambda = 0$ this ODE has a PF in μ. Are interested in a two parameter (μ, λ) characterization of all possible bifurcations.

Find the locus of nonhyperbolic points

$$\left.\begin{array}{lll} (1) & f = \lambda + \mu x - x^3 = 0 \\ (2) & f_x = \mu - 3x^2 = 0 \end{array}\right\} \text{eliminate } x \text{ from these.}$$

Solve (2) to get

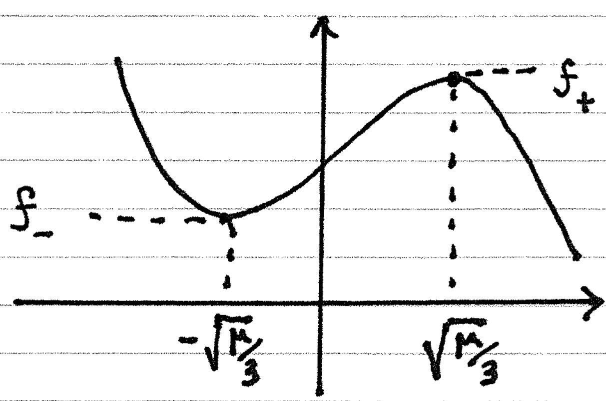
$$x = \pm\sqrt{\mu/3} \qquad \mu > 0$$

Use this in (1) and solve $f = 0$ for λ

$$(3) \qquad \lambda = \pm 2\left(\frac{\mu}{3}\right)^{3/2}$$

locus of nonhyperbolic points

Can deduce shape of f as follows.



$$f_\pm \equiv \lambda \pm 2\left(\frac{\mu}{3}\right)^{3/2}$$

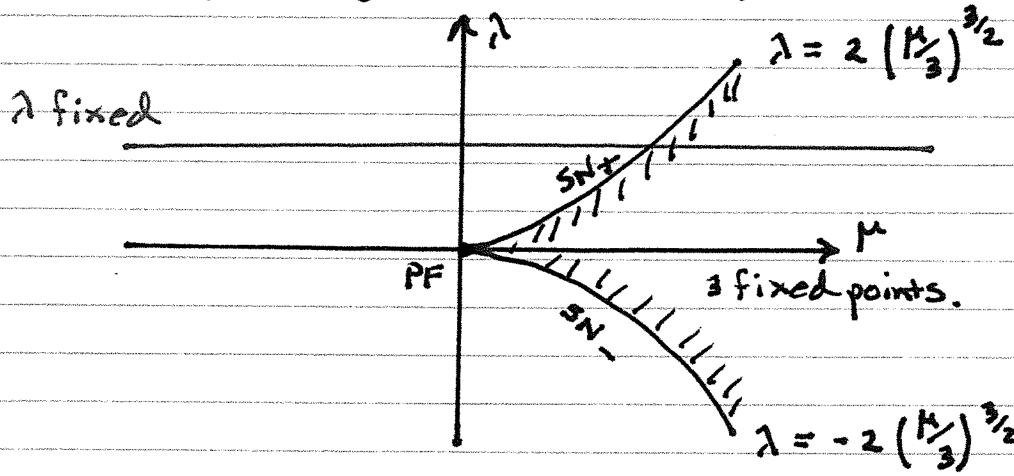
Can have 3 fixed points only if

i) $f_+ > 0$

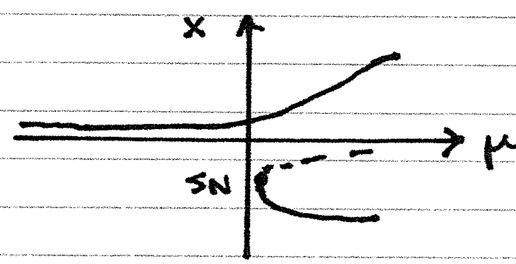
ii) $f_- < 0$

iii) $\mu > 0$

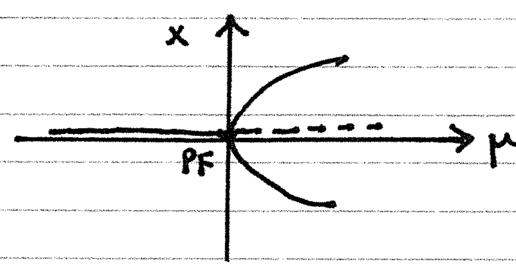
Stability Diagram in (μ, λ)-plane



We haven't proven the SN bifurcations exist but graphs of λ fixed cross sections can show this

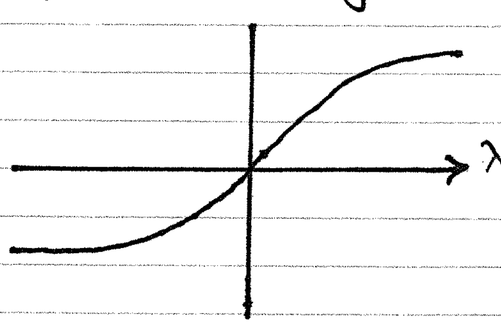


$\lambda > 0$

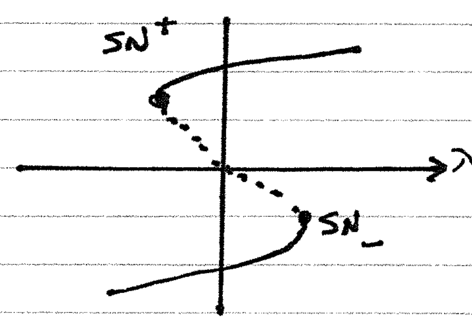


$\lambda = 0$

For μ-fixed diagrams



$\mu < 0$



$\mu > 0$

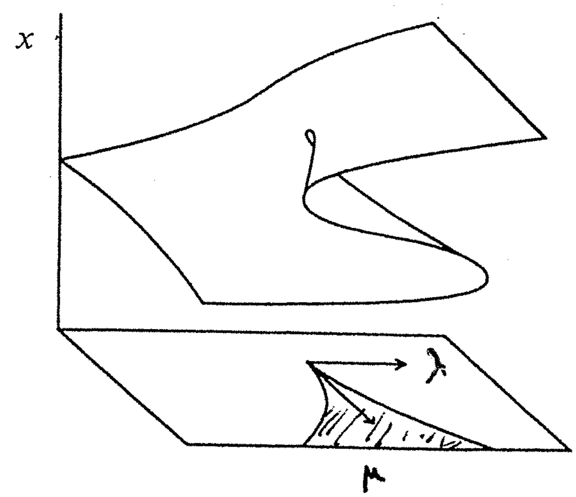


Figure 3.6.5

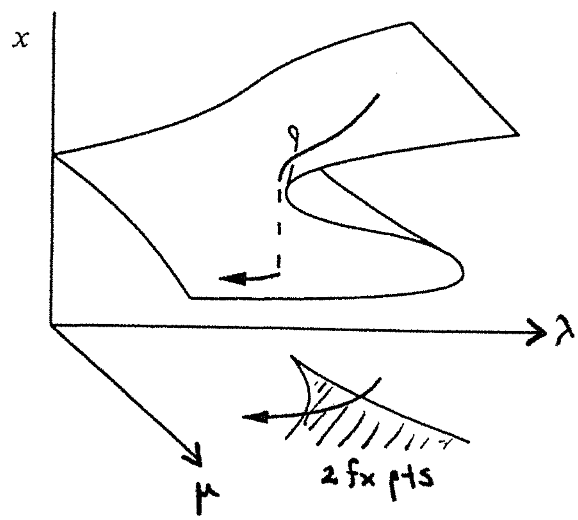
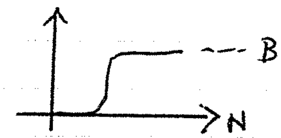


Figure 3.6.6

Budworm Model

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) \quad - p(N)$$

$$p(N) = \frac{BN^2}{A^2 + N^2} \quad \text{predation}$$



"not so steep"

Nondimensional model

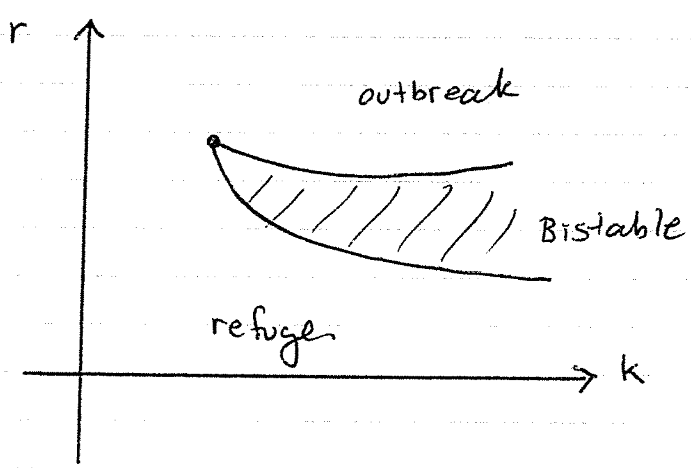
$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{K}\right) - \frac{x^2}{1 + x^2} = f(x, r, K)$$

Conditions for nonhyperbolic

$$\left.\begin{array}{l} f(x, r, K) = 0 \\ f_x(x, r, K) = 0 \end{array}\right\} \quad \begin{array}{ll} \text{solve for} & r = r(x) \\ \quad \text{then} & k = k(x) \end{array}$$

$$r(x) = \frac{2x^3}{(1 + x^2)^2}$$

$$k(x) = \frac{2x^3}{x^2 - 1} > 0 \quad \Rightarrow \quad x > 1$$



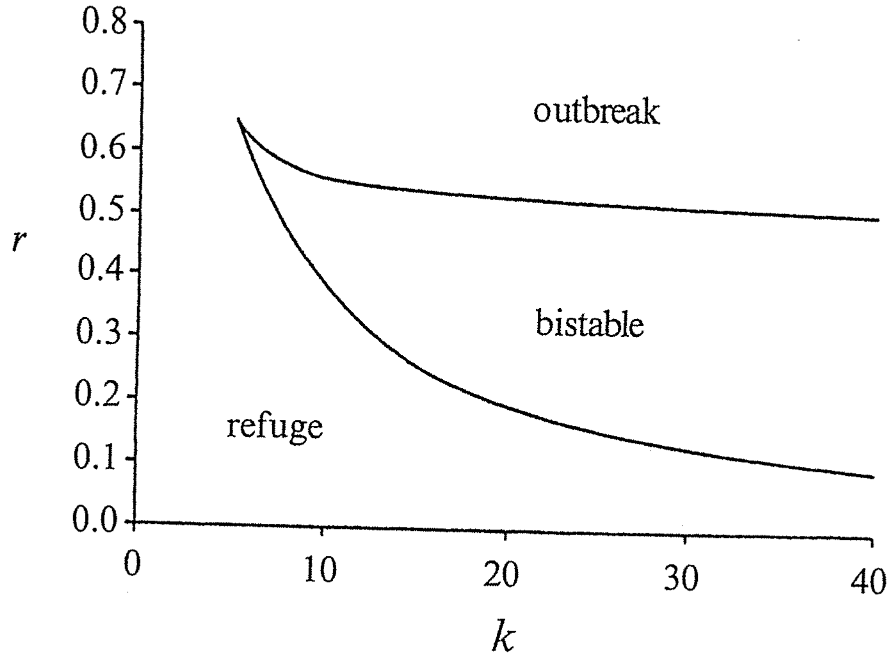


Figure 3.7.5

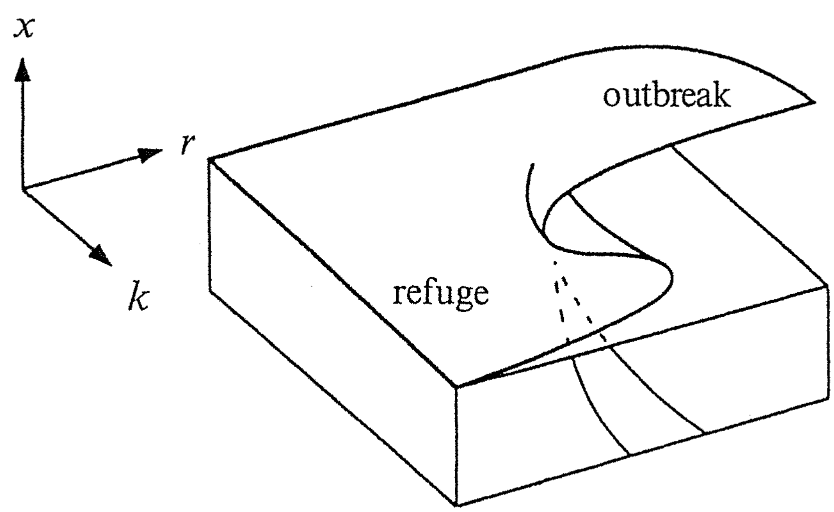


Figure 3.7.6