Dynamics on $\mathbb{R}$

$$\dot{x} = f(x) \qquad\qquad f : \mathbb{R} \to \mathbb{R}$$

Here f is a <u>vector field on $\mathbb{R}$</u>.

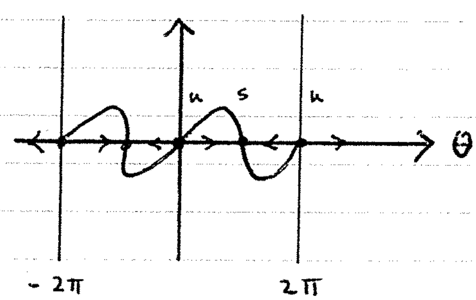
Dynamics on S^1

Consider $f : \mathbb{R} \to \mathbb{R}$ where f is 2π-periodic:

$$f(\theta + 2\pi) = f(\theta) \qquad\qquad \forall \theta \in \mathbb{R}.$$

and associated differential equation

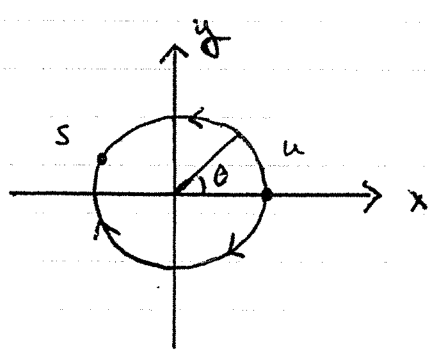
$$\dot{\theta} = f(\theta)$$



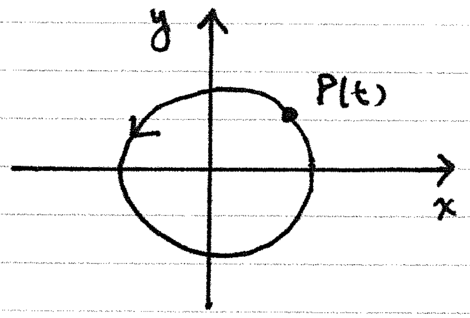
Dynamics repeats

Can consider dynamics as vector field on S^1 where

$$S^1 = \text{unit circle in } \mathbb{R}^2$$



Periodic solutions on S^1



If $f(\theta)$ is of <u>one sign</u> on $[0, 2\pi]$ the system

$$(1) \qquad \dot{\theta} = f(\theta)$$

describes periodic motion on S^1. In this case one can compute the period T. Easy since (1) is separable

$$\boxed{T = \int_0^{2\pi} \frac{d\theta}{f(\theta)}}$$

EXAMPLE Pure rotation with angular velocity ω

$$\dot{\theta} = \omega \qquad\qquad T = \frac{2\pi}{\omega}$$

EXAMPLE Dynamics on S^1 with

$$\dot{\theta} = 1 + \cos^2\theta > 0$$

Period is a hard but doable integral

$$T = \int_0^{2\pi} \frac{d\theta}{1 + \cos^2\theta} = \frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}} \tan\theta\right)\Bigg|_0^{2\pi}$$

$$T = \sqrt{2}\,\pi$$

Bifurcations on S^1 (example)

$$\boxed{\begin{aligned}\dot\theta &= \mu\sin\theta - \sin 2\theta \\ \dot\theta &= \sin\theta\,(\mu - 2\cos\theta)\end{aligned}}$$

trigident

Fixed points: $\bar\theta = 0, \pi$ and $\bar\mu(\theta) = 2\cos\theta$

Linear stability of $\bar\theta = 0$ yields $f'(0) = \mu - 2 < 0$ only if $\mu < 2$ (stable).

Nonhyperbolic points found in usual way

$$f = \mu\sin\theta - \sin 2\theta = 0$$

$$f' = \mu\cos\theta - 2\cos 2\theta = 0$$

has two solutions $(\theta^*, \mu^*) = (0, 2)$ and $(\pi, -2)$

Collectively