1 Planar Systems - Preliminary Definitions

By a planar system of differential equations we mean a system of the form:

$$\begin{aligned} \dot{x_1} &= f_1(x_1, x_2) \\ \dot{x_2} &= f_2(x_1, x_2) \end{aligned}$$

where $f_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$. This system can be written in the compact form

$$\dot{x} = f(x) \tag{1}$$

by making the identifications:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad , \quad f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \quad .$$

In particular, f is a vector-valued function, i.e., $f : \mathbb{R}^2 \to \mathbb{R}^2$. Also, we may write the column vector x above as $x = (x_1, x_2)^T$ where the superscript ^{T} means transpose. Unless otherwise stated we will assume that solutions exist for all time and that the components of f are twice continuously differentiable on $\mathbb{R}^2$. To make latter definitions more compact we define the Euclidean norm of $x \in \mathbb{R}^2$ by

$$\| x \| = \sqrt{x_1^2 + x_2^2} \quad , \quad x = (x_1, x_2)^T \quad .$$

Then the Euclidean distance between $x, y \in \mathbb{R}^2$ is

$$d(x, y) = \| x - y \| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

We shall then define a δ-neighbourhood of x as the set of points a distance at most δ from x, or:

$$N_\delta(x) = \{ y \in \mathbb{R}^2 \ : \| x - y \| < \delta . \}$$

Definition 1 *$\bar{x}$ is a fixed point of $\dot{x} = f(x)$, $f : \mathbb{R}^2 \to \mathbb{R}^2$, if $f(\bar{x}) = 0$.*

Definition 2 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is isolated if $\exists \delta > 0$ such that $y \in N_\delta(\bar{x})$ and $y \neq x$ implies $f(y) \neq 0$.*

Thus, if $\bar{x}$ is an isolated fixed point, there is some (small) neighbourhood of $\bar{x}$ that contains no other fixed point. We will adopt the following definitions related to the stability of fixed points.

Example The system

$$\begin{aligned}\dot{x} &= y - 1 \\ \dot{y} &= x - 2\end{aligned}$$

has a sole isolated fixed point $\bar{\mathbf{x}} = (2, 1)$.

Example The system

$$\begin{aligned}\dot{x} &= x - y \\ \dot{y} &= 2x - 2y\end{aligned}$$

has a line of fixed points $\bar{\mathbf{x}} = (x, x), x \in \mathbb{R}$. None of them are isolated. every neighbourhood of any point on this line ($y = x$) contains other points on it.

Example The system

$$\begin{aligned}\dot{x} &= y - x^2 \\ \dot{y} &= y - 4\end{aligned}$$

has two isolated fixed points. Their coordinates must satisfy the two simultaneous equations:

$$\begin{aligned}y - x^2 &= 0 \\ y - 4 &= 0\end{aligned}$$

leading to the two fixed points $\bar{\mathbf{x}}_1 = (2, 4)$ and $\bar{\mathbf{x}}_2 = (-2, 4)$

Example The system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where the matrix A is invertible has only the origin as a fixed point since

$$A\mathbf{x} = 0 \Rightarrow \mathbf{x} = A^{-1}0 = 0$$

Example The system

$$\begin{aligned}\dot{x} &= x(y - x^3) \\ \dot{y} &= y(x - y)\end{aligned}$$

has three fixed points. By inspection $\bar{\mathbf{x}}_0 = (0, 0)$ is one. At the others $y = x^3$ so the second equation implies

$$x - x^3 = 0$$

hence $x = 0, \pm 1$ and $\bar{\mathbf{x}}_0 = (1, 1)$, $\bar{\mathbf{x}}_0 = (-1, -1)$

Definition 3 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is Liapunov stable if $\forall \epsilon > 0 \; \exists \delta > 0$ such that if $x(t)$ is a solution of*

$$\dot{x} = f(x) \quad , \quad x(t_0) = x_0 \in N_\delta(\bar{x})$$

then

$$\| x(t) - \bar{x} \| < \epsilon \quad , \quad \forall t \geq t_0 \, .$$

Notice that for this definition to make sense $\delta \leq \epsilon$ else there would be some initial conditions $x_0 \in N_\delta(\bar{x})$ for which $x(t)$ would initially be outside the neighbourhood $N_\epsilon(\bar{x})$ it is required to remain in for all $t \geq t_0$. In words, this definition implies that the solutions $x(t)$ remain close to the fixed point for all time if the initial condition is sufficiently close to $\bar{x}$.

Definition 4 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is stable if it is Liapunov stable.*

Thus, for our conventions, stable and Liapunov stable are equivalent.

Definition 5 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is attracting if $\exists \delta > 0$ such that if $x(t)$ is a solution of*

$$\dot{x} = f(x) \quad , \quad x(t_0) = x_0 \in N_\delta(\bar{x})$$

then

$$\lim_{t \to \infty} \| x(t) - \bar{x} \| = 0$$

Notice that this definition does not preclude the possibility of $x(t)$ leaving the neighbourhood $N_\delta(\bar{x})$ for some time. However, it does imply that $x(t)$ must eventually return to and stay in $N_\delta(\bar{x})$. Since large excursions are possible, some attracting fixed points are normally not thought of as "stable".

Definition 6 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is globally attracting if $\forall x_0 \in \mathbb{R}^2$ the solution $x(t)$ of*

$$\dot{x} = f(x) \quad , \quad x(t_0) = x_0$$

satisfies

$$\lim_{t \to \infty} \| x(t) - \bar{x} \| = 0.$$

Definition 7 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is asymptotically stable if it is both Liapunov stable and attracting.*

Definition 8 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is neutrally stable if it is Liapunov stable but not attracting.*

Definition 9 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is* unstable *if it is not stable.*

Notice that, by definition, if $\bar{x}$ is asymptotically stable it is also Liapunov stable. Also, not all attracting fixed points are necessarily asymptotically stable so that the definitions are mutually exclusive. Below we summarize these definitions in a Table:

	Attracting?	Liapunov Stable?
Asymptotic Stability	Y	Y
Neutral Stability	N	Y
Unstable	N	N
	Y	N

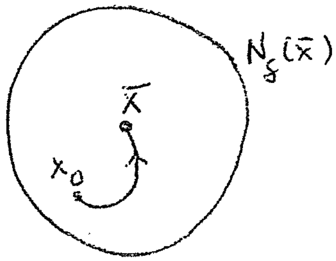
It should be remarked that there are different conventions in the definitions of stability throughout the literature. Some also address stability of solutions that are not fixed points. For instance, some authors define the Liapunov stability of a solution $x^*(t)$ of an initial value problem in the same way that the Liapunov stability of a fixed point is defined. In particular,

$$x_0 \in N_\delta(x^*(t_0)) \Rightarrow \| x(t) - x^*(t) \| < \epsilon \quad , \quad \forall t \geq t_0$$

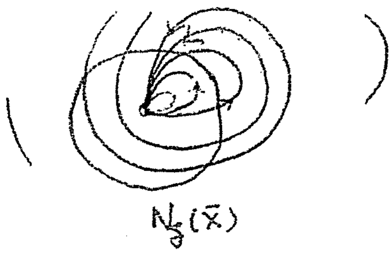
This sort of issue will become important when we talk about periodic solutions of planar systems.

More importantly, depending on the source the definitions above can have different names. For example, in [7], "Liapunov Stable" is "stable" but our definition of "attracting" is equivalent to their definition of "asymptotically stable" (page 128). In our textbook [8], "stable" and "asymptotically stable" are equivalent which should be contrasted with [5] (page 266) where "stable" is equivalent to "Liapunov stable" (the convention we adopt). The author [9] has identical definitions to ours except do not have a definition for an "attracting" fixed point. In [5], "attracting" fixed points are also not defined. In [4, 6], there are also separate definitions for uniform stability and Poincare stability. Thus, when consulting other resources it is important to know which definition is being used! The definitions for "stability" are especially important since "unstable" is most often defined as "not stable". Our textbook, however, gives the definition of "unstable" as "neither attracting nor Liapunov stable" (page 129). This definition is a bit vague in my opinion. If one is to interpret that as meaning it is not Liapunov stable and it is not attracting then a fixed point that is attracting but but not Liapunov stable is not stable (by their definition) yet it is not unstable! For this and other reasons we are adopting the definitions set out in this writeup.

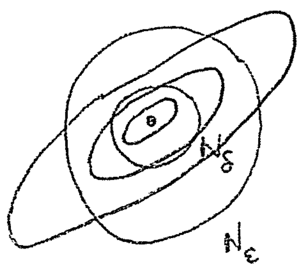
Pictures for Stability



asymptotically stable

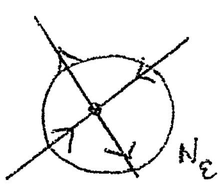


attracting (but not Liapunov)
(not asymptotic)



Liapunov but not asymptotic

Neutral Stability



Unstable

NOT Liapunov $\iff$ $\forall \varepsilon > 0$ $\nexists \delta > 0$ s.t.
$\|x(t) - \bar{x}\| < \varepsilon$, $\forall t \geq 0$.
and all $x_0 \in N_\delta(\bar{x})$

Hence $\exists x_0$ in $N_\varepsilon(\bar{x})$ s.t. $x(\bar{t}; x_0) \notin N_\varepsilon(\bar{x})$
for some $\bar{t} > t_0$.