Math 472: Homework 4
Due: Friday, April 1, 2016.

Instructions: In every problem sketch the integration curve and indicate the location of all integrand singularities (with the exception of 1. which is not analytic anywhere).

1. [5] Evaluate
\[ I = \int_C \frac{\bar{z}}{z} \, dz \]
where \( C \) is the straight line from \( z_1 = 1 \) to \( z_2 = 3i \). Note that \( C \) is given by:
\[ z(t) = (1 - t)z_1 + tz_2, \quad t \in [0, 1] \]

2. [5] Let \( C \) be the unit semicircle \( z = e^{i\theta} \) that crosses the cut of Principal \( \text{Log}(z) \). The ranges of \( \theta \) are \( \theta \in \left[ \frac{\pi}{2}, \pi^- \right] \) and \( \theta \in \left[ -\pi^+, -\frac{\pi}{2} \right] \). Evaluate
\[ I = \int_C \text{Log}(z) \, dz \]
directly using the parametrization \( z = e^{i\theta} \). Remember to split the integral into two parts.

3. [5] Let
\[ f(z) \equiv \frac{z}{\sqrt{z^2 + 1}} \]
Draw the cut \( \Gamma \) of \( f(z) \) and any curve \( C \) (not crossing \( \Gamma \)) from \( z_1 = 2 + i \) to \( z_2 = -2 \). Then use an antiderivative of \( f(z) \) to evaluate
\[ I = \int_C f(z) \, dz \]

4. [5] Evaluate
\[ I = \oint_C \frac{e^z}{z - 2} \, dz \]
where (a) \( C \) is the circle \(|z| = 3\) and (b) \( C \) is the circle \(|z| = 1\).
5. [5] Evaluate
\[ I_n \equiv \oint_{|z|=1} \frac{\sin 2z}{z^n} \, dz \]
for \( n = 0, 1, 2, 3, 4 \).

6. [5] Use partial fraction expansions to evaluate
\[ \oint_{|z|=5} \frac{2z + 1}{z^2 + 3z + 2} \, dz \]

7. [5] Prove the real integral
\[ \int_0^\pi e^{\cos \theta} \cos (\sin \theta) \, d\theta = \pi \]
by considering the real/imaginary part of the complex integral
\[ \oint_{|z|=1} \frac{e^z}{z} \, dz \]

8. [5] Evaluate
\[ \oint_C \frac{dz}{(z^2 + 1)(z^2 + 4)} \]
where \( C \) is the upper \((\text{Im}(z) > 0)\) closed semicircle of radius 3.