Defn The system of complex numbers $\mathbb{C}$ is the set $\mathbb{R}^2$ with the following addition and multiplication operations for $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

(1) $$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$

(2) $$a z_1 = (a x_1, a y_1) \qquad \forall a \in \mathbb{R}$$

(3) $$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Although there is not real number i such that $i^2 = -1$ there is a complex number.

Defn The complex number $i \in \mathbb{C}$ is the ordered pair

$$i \equiv (0, 1)$$

Remarks and notation

Given the defn (3) above

$$i^2 = i \cdot i = (0, 1)(0, 1) = (-1, 0)$$

from which we have the identification

$$i^2 = -1$$

$\boxed{\textbf{Theorem} \quad \mathbb{C} \text{ is a field}}$

We verify only a few axioms

Multiplication commutes $z_1 z_2 = z_2 z_1$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$
$$= (x_2 x_1 - y_2 y_1,\ x_2 y_1 + x_1 y_2)$$
$$= z_2 z_1$$

Existence of additive identity $z + 0 = z$

The number $0 \in \mathbb{C}$ is the point $0 = (0,0)$. Then

$$z + 0 = (x,y) + (0,0) = (x,y) = z$$

Existence of multiplicative identity $1 \cdot z = z$

The number $1 \in \mathbb{C}$ is the point $\underline{1} = (1,0)$:

$$1 \cdot z = (1,0) \cdot (x,y) = (x,y) = z$$

For any real number $a \in \mathbb{R}$

$$az = (a,0) \cdot (x,y) = (ax, ay)$$

Multiplicative inverse z^{-1}

Let $z = (x, y) \neq 0$. Seek $u, v \in \mathbb{R}$ with $z^{-1} = (u, v)$ such that

$$z \cdot z^{-1} = 1$$

is equivalent to

$$(1) \qquad (x, y)(u, v) = (1, 0)$$

Expanding out (1) using the definition of multiplication and equating components:

$$\left.\begin{aligned} xu - yv &= 1 \\ yu + xv &= 0 \end{aligned}\right\} \text{ solve for } u, v$$

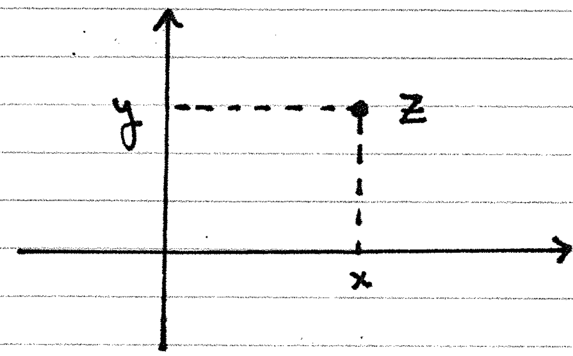
Solving

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$$

For this (unique) z^{-1} we have $z \cdot z^{-1} = 1$.

Complex Plane and simplified notation

Every complex number $z=(x,y)$ corresponds to a point in $\mathbb{R}^2$



and has a real and imaginary part

$$\mathbb{R}e(z) = x \qquad \mathbb{I}m(z) = y$$

Simplified notation

$$z = (x, y)$$

$$(2) \qquad z = x\underbrace{(1,0)}_{1} + y\underbrace{(0,1)}_{i}$$

Hence we abbreviate (2) as

$$z = x + iy$$

Multiplication using simplified notation

$$z_1 z_2 = (x_1 + i y_1)(x_2 + i y_2)$$

$$= x_1 (x_2 + i y_2) + i y_1 (x_2 + i y_2)$$

$$= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + y_1 y_2 i^2$$

Given $i = (0,1)$ and $i^2 = (-1,0)$ we get

$$= (x_1 x_2 - y_1 y_2 , x_1 y_2 + x_2 y_1)$$

which agrees with the original defn of multiplication.

Multiplicative inverse - simplified notation

$$z^{-1} = \frac{1}{(x+iy)} \underbrace{\frac{(x-iy)}{(x-iy)}}_{1}$$

$$z^{-1} = \frac{x - iy}{x^2 + y^2}$$

agrees with our previous formula for z^{-1}

$$\boxed{\begin{array}{l}\underline{\text{Defn}} \quad \text{Let } z, w \in \mathbb{C}. \text{ Then the quotient is defined by} \\ \qquad \dfrac{z}{w} \equiv z \cdot w^{-1} \qquad w \neq 0\end{array}}$$

<u>EXAMPLE</u> Let $z = 1+2i$ and $\omega = 1-3i$

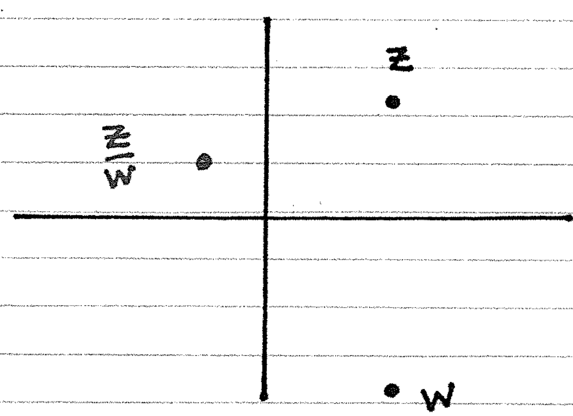
$$\frac{z}{w} = (1+2i) \frac{1}{(1-3i)} \frac{(1+3i)}{(1+3i)}$$

$$= (1+2i) \frac{(1+3i)}{10}$$

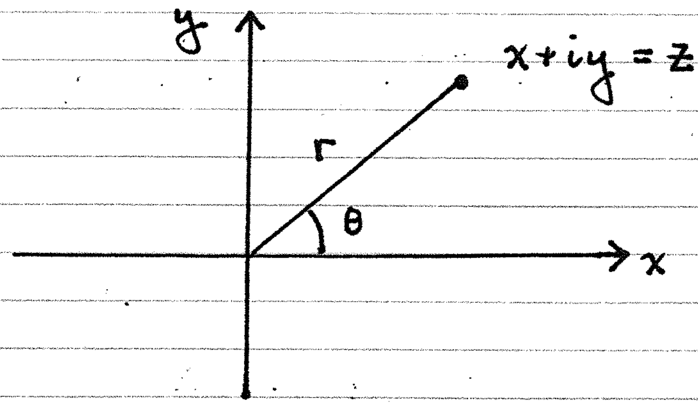
$$= (1+2i)\left(\frac{1}{10} + \frac{3}{10}i\right)$$

$$= -\frac{5}{10} + \frac{5}{10}i$$

$$= -\frac{1}{2} + \frac{1}{2}i$$



Polar Representation



$r > 0$

$\theta \in I$

Specification of an interval I so that the correspondence $(x,y) \leftrightarrow (r,\theta)$ is unique is known as choosing a <u>branch</u> of the <u>argument</u>

$$I = (-\pi, \pi]$$

yields the Principal Argument $\theta = \operatorname{Arg} z$

<u>Definition</u> Let $z = x+iy \in \mathbb{C}$. Then

(i) $|z| \equiv \sqrt{x^2+y^2}$ modulus

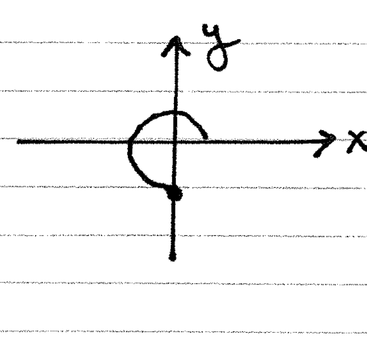
(ii) $\theta = \arg z$ $\theta \in I$, argument

<u>EXAMPLE</u> Let $\arg(z) \in [0, 2\pi)$ and $\operatorname{Arg}(z) \in [-\pi, \pi)$

If $z = -i$ then

$$\arg(z) = \frac{3\pi}{2}$$

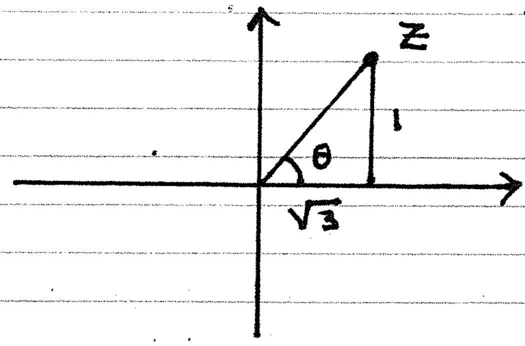
$$\operatorname{Arg}(z) = -\frac{\pi}{2}$$



EXAMPLE Let $z = \dfrac{1 + 3\sqrt{3}\, i}{\sqrt{3} + 2i}$

After some calculations $z = \sqrt{3} + i$

$$\mathrm{Re}(z) = \sqrt{3} \qquad \mathrm{Im}(z) = 1$$



From this it is easy to deduce (for $I = [0, 2\pi)$)

$$|z| = 2 \qquad \arg z = \frac{\pi}{6}$$

EXAMPLE Compute real and imaginary parts of $f(z)$ for $z = x + iy$

(i) $f(z) \equiv \dfrac{1}{2z+1} = \dfrac{(2x+1) - 2iy}{(2x+1)^2 + 4y^2}$

(ii) $f(z) \equiv z^2 = (x^2 - y^2) + 2ixy$

(iii) $f(z) \equiv |z| = \sqrt{x^2 + y^2}$

EXAMPLE: Simplify

$$z = \frac{i^3}{(1+i)}$$