The system of complex numbers \( \mathbb{C} \) is the set \( \mathbb{R}^2 \) with the following addition and multiplication operations for \( z_1 = (x_1, y_1) \) and \( z_2 = (x_2, y_2) \):

1. \( z_1 + z_2 = (x_1 + x_2, y_1 + y_2) \)
2. \( az_1 = (ax_1, ay_1) \quad \forall a \in \mathbb{R} \)
3. \( z_1 z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \)

Although there is not real number \( i \) such that \( i^2 = -1 \), there is a complex number.

The complex number \( i \in \mathbb{C} \) is the ordered pair

\[ i = (0, 1) \]

Remarks and notation

Given the defn (3) above

\[ i^2 = i \cdot i = (0, 1)(0, 1) = (-1, 0) \]

from which we have the identification

\[ i^2 = -1 \]
Theorem: \( \mathbb{C} \) is a field

We verify only a few axioms:

**Multiplication commutes** \( z_1 z_2 = z_2 z_1 \)

\[
\begin{align*}
z_1 z_2 &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \\
&= (x_2 x_1 - y_2 y_1, x_2 y_1 + x_1 y_2) \\
&= z_2 z_1
\end{align*}
\]

**Existence of additive identity** \( z + 0 = z \)

The number 0 \( \in \mathbb{C} \) is the point \( 0 = (0,0) \). Then

\[
z + 0 = (x,y) + (0,0) = (x,y) = z
\]

**Existence of multiplicative identity** \( 1 \cdot z = z \)

The number 1 \( \in \mathbb{C} \) is the point \( 1 = (1,0) \):

\[
1 \cdot z = (1,0) \cdot (x,y) = (x,y) = z
\]

For any real number \( a \in \mathbb{R} \)

\[
a z = (a,0) \cdot (x,y) = (ax,ay)
\]
Multiplicative inverse $z^{-1}$

Let $z = (x, y) \neq 0$. Seek $u, v \in \mathbb{R}$ with $z^{-1} = (u, v)$ such that

$$z \cdot z^{-1} = 1$$

is equivalent to

$$(1) \quad (x, y)(u, v) = (1, 0)$$

Expanding out (1) using the definition of multiplication and equating components:

$$xu - yv = 1$$
$$yu + xv = 0$$

Solving

$$z^{-1} = \left( \frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right)$$

For this (unique) $z^{-1}$ we have $z \cdot z^{-1} = 1$. 
Complex plane and simplified notation

Every complex number \( z = (x, y) \) corresponds to a point in \( \mathbb{R}^2 \)

\[ y \quad \ldots \quad z \]

and has a real and imaginary part

\[ \text{Re}(z) = x \quad \text{Im}(z) = y \]

Simplified notation

\[ z = (x, y) \]

(2) \[ z = x(1, 0) + y(0, 1) \]

Hence we abbreviate (2) as

\[ z = x + iy \]
Multiplication using simplified notation

\[ z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) \]
\[ = x_1(x_2 + iy_2) + iy_1(x_2 + iy_2) \]
\[ = x_1 x_2 + i x_1 y_2 + i y_1 x_2 + y_1 y_2 i^2 \]

Given \( i = (0,1) \) and \( i^2 = (-1,0) \) we get

\[ = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1) \]

which agrees with the original defn of multiplication.

Multiplicative inverse - simplified notation

\[ z^{-1} = \frac{1}{(x+iy)(x-iy)} \]
\[ = \frac{x - iy}{x^2 + y^2} \]

agrees with our previous formula for \( z^{-1} \)
Let $z, w \in \mathbb{C}$. Then the quotient is defined by

$$\frac{z}{w} = z \cdot w^{-1} \quad w \neq 0$$

**Example**

Let $z = 1+2i$ and $w = 1-3i$

$$\frac{z}{w} = \frac{(1+2i)}{1-3i} \cdot \frac{1}{1+3i}$$

$$= (1+2i) \cdot \frac{1+3i}{10}$$

$$= (1+2i) \cdot \left(\frac{1}{10} + \frac{3}{10}i\right)$$

$$= -\frac{5}{10} + \frac{5}{10}i$$

$$= -\frac{1}{2} + \frac{1}{2}i$$
Polar Representation

\[ x + iy = z \quad r > 0 \]
\[ \theta \in \mathbb{I} \]

Specification of an interval \( \mathbb{I} \) so that the correspondence \((x, y) \leftrightarrow (r, \theta)\) is unique is known as choosing a branch of the argument.

\[ \mathbb{I} = (-\pi, \pi] \]

yields the Principal Argument \( \theta = \text{Arg } z \)

Definition: Let \( z = x + iy \in \mathbb{C} \). Then

(i) \( |z| = \sqrt{x^2 + y^2} \) modulus

(ii) \( \theta = \text{arg } z \quad \theta \in \mathbb{I} \), argument

**Example**: Let \( \text{arg}(z) \in [0, 2\pi) \) and \( \text{Arg}(z) \in (-\pi, \pi) \)

If \( z = -i \) then

\[ \text{arg}(z) = \frac{3\pi}{2} \]
\[ \text{Arg}(z) = -\frac{\pi}{2} \]
**EXAMPLE**  \( \text{Let } z = \frac{1 + 3\sqrt{3}i}{\sqrt{3} + 2i} \)

After some calculations, \( z = \sqrt{3} + i \)

\[ \text{Re}(z) = \sqrt{3}, \quad \text{Im}(z) = 1 \]

From this it is easy to deduce (for \( I = [0, 2\pi] \))

\[ |z| = 2, \quad \arg z = \frac{\pi}{6} \]

**EXAMPLE**  Compute real and imaginary parts of \( f(z) \) for \( z = x + iy \)

(i) \[ f(z) = \frac{1}{z + 1} = \frac{(2x + 1) - 2iy}{(2x + 1)^2 + 4y^2} \]

(ii) \[ f(z) = z^2 = (x^2 - y^2) + 2ixy \]

(iii) \[ f(z) = |z|^2 = \sqrt{x^2 + y^2} \]

**EXAMPLE:** Simplify

\[ z = \frac{i^3}{(1 + i)} \]