Complex Contours

\[ C \]

\[ z(t) = x(t) + iy(t) \quad t \in [a, b] \]

Curves can be continuous, differentiable, or piecewise differentiable depending on whether \( x(t) \) and \( y(t) \) are.

**Defn:** A curve \( C \) is simple if it does not cross itself:

\[ z(t_1) \neq z(t_2) \quad \forall t_1 \neq t_2 \in (a, b) \]

**Defn:** A simple closed curve \( C \) is a simple curve with \( z(a) = z(b) \)

**Defn:** A closed curve is positively oriented if it traverses counter clockwise
Contour Terminology

- Simple
- Simple closed
- Positively oriented
- Not simple

- Piecewise differentiable
- Continuous simple closed, negative orientation
Contour Integral Definition

\[ C \]

\[ t = b \]

\[ t = a \]

\[ \Delta Z_i \]

\[ Z(t) = x(t) + iy(t) \]

Riemann sum definition

\[ \Delta Z_i = Z(t_i^+) - Z(t_i^-) \quad i = 1, 2, \ldots, N \]

then

\[ \int_C f(z) \, dz \equiv \lim_{\Delta z_i \to 0} \sum_{i=1}^{N} f(z_i^+) \Delta z_i \]

is a line integral of \( f(z) \) along curve \( C \) in the complex plane.

Not proven here is the following fact (for differentiable curves \( C \))

(1) \[ \int_C f(z) \, dz = \int_a^b f(z(t)) \, z'(t) \, dt \]
Alternate Notations

\[ f(z) = u(x, y) + iv(x, y) \]

Given \( z(t) = x(t) + iy(t) \) on curve \( C \) the integral \((1)\) can be written

\[
\int_C f(z)\,dz = \int_a^b (u(t) + iv(t))(x'(t) + iy'(t))\,dt
\]

which can be separated into real and imaginary parts.

\[
\int_C f(z)\,dz = \int_a^b (ux' - vy')\,dt + i\int_a^b (vy' + vx')\,dt
\]

Eliminating \( t \) we arrive at the conclusion

\[
\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy
\]
EXAMPLE

\[ \oint_C \overline{z} \, dz \quad \text{on} \quad z(t) = 3e^{it}, \quad t \in [0, \frac{\pi}{2}] \]

\[ I = \oint_C \overline{z} \, dz \]

\[ I = \int_{0}^{\frac{\pi}{2}} \overline{z(t)} z'(t) \, dt \]

\[ I = \int_{0}^{\frac{\pi}{2}} (3e^{-it})(3ie^{it}) \, dt \]

\[ I = \int_{0}^{\frac{\pi}{2}} 9i \, dt \]

\[ I = \frac{9\pi}{2}i \]

radius 3

quarter circle
EXAMPLE
\[ \oint_C \frac{dz}{z-1} \] where \( C \) is the counterclockwise circle of radius 1 centered at \((1,0)\)

First need to parametrize the circle

\[ z = 1 + e^{it} \quad t \in [0, 2\pi] \]
\[ z' = ie^{it} \]

\[ \oint_C \frac{dz}{z-1} = \int_0^{2\pi} \frac{1}{(1+e^{it})-1} (ie^{it}) \, dt \]
\[ = \int_0^{2\pi} i \, dt \]
\[ = 2\pi i \]

Note the integral is well defined despite the singularity in the center of \( C \)

\[ f(z) = \frac{1}{z-1} \quad z \neq 1 \]
EXAMPLE \[ \oint_{C_1} z^2 \, dz \] where \( C_1 \) is the straight line from \((0,0)\) to \((1,1)\).

\[ z(t) = t + t \, i \quad t \in [0,1] \]

\[ z'(t) = (1+i) \]

After some calculations

\[ I = \int_0^1 (t+t \, i)^2 (1+i) \, dt = \frac{2}{3} (i-1) \]

EXAMPLE Compute \( \oint_{C_2} z^2 \, dz \) on \( z(t) = t + t^2 \, i \) parabola!

\[ I = \int_0^1 (t+t^2 \, i)(1+2t \, i) \, dt = \frac{2}{3} (i-1) \quad \text{same!} \]

EXAMPLE Show \( \oint_{C_1 \cup C_2} z^2 \, dz = 0 \). Use previous two examples

\[ I = \oint_{C_2} z^2 \, dz - \oint_{C_1} z^2 \, dz = 0 \]
EXAMPLE

\[ \oint_C \bar{z} \, dz \]

where \( C \) is the line segment from 3 to 3i

\[ Z(t) = 3(1-t) + 3it \quad t \in [0,1] \]

Note that \( Z(0) = 3, \quad Z(1) = 3i \)

\[ Z'(t) = -3 + 3i \]

\[ \oint_C \bar{z} \, dz = \int_0^1 (x(t) - iy(t)) Z'(t) \, dt \]

\[ = \int_0^1 (3(1-t) - 3it)(-3+3i) \, dt \]

\[ = \left( \frac{3}{2} - \frac{3}{2}i \right)(-3+3i) = 9i \]

This should be contrasted with previous \( \oint_C \bar{z} \, dz \)

on quarter circle

\[ \oint_C \bar{z} \, dz = \frac{9\pi}{2}i \]

\[ \oint_C \bar{z} \, dz = 9i \]

Path dependent result (since \( \bar{z} \) not analytic)
EXAMPLE

\[ \int_{C} \frac{dz}{z^n} \text{ on unit circle} \]

\[
= \int_{0}^{2\pi} \frac{1}{e^{n\pi i}} (i e^{it}) \, dt \\
= \int_{0}^{2\pi} i e^{i(1-n)t} \, dt \\
= \begin{cases} 
2\pi i & \text{n = 1} \\
0 & \text{n \neq 0} 
\end{cases}
\]

EXAMPLE

\[ \int_{C} 1z^{2} \, dz \text{ on unit circle} \]

On C, \( z(t) = e^{it} \) and \( |z(t)|^2 = 1 \) for all \( t \).

\[
\int_{C} |z|^2 \, dz = \int_{0}^{2\pi} (1)^2 \cdot i e^{it} \, dt \\
= \left. e^{it} \right|_{0}^{2\pi} \\
= 0
\]
EXAMPLE

Evaluate \( \int_C \frac{dz}{z(z^2-1)} \) where \( C \)below

\[ \begin{align*}
  z(t) &= e^{i\theta} \quad \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{2}\right) \\
  z'(t) &= i e^{i\theta} \\
  \end{align*} \]

\[ \int_C \frac{dz}{z(z^2-1)} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{ie^{i\theta}}{e^{i\theta}(e^{2i\theta} - 1)} d\theta \frac{e^{-i\theta}}{e^{-i\theta}} \]

\[ = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} \frac{e^{-i\theta}}{\sin\theta} d\theta \]

\[ = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{2}} (\cot\theta - i) d\theta \]

\[ = \frac{1}{2} \ln(\sin\theta) \bigg|_{\frac{\pi}{4}}^{\frac{3\pi}{2}} - \frac{\pi}{8} i \]

\[ = \frac{1}{4} \ln 2 - \frac{\pi}{8} i \]
EXAMPLE Integrating over a Branch Cut

Moral: Take correct value on either side of the branch

\[ \int_{C} z^{1/2} \, dz = \int_{C_{+}} z^{1/2} \, dz + \int_{C_{-}} z^{1/2} \, dz \]

Above is a portion of the unit circle

On \( C_{+} \) \quad \( z = e^{i\theta} \Rightarrow \sqrt{z} = e^{i\theta/2} \) only if \( \theta \in [\pi, \pi] \)

On \( C_{-} \) \quad \( z = e^{i\theta} \Rightarrow \sqrt{z} = e^{i\theta/2} \) only if \( \theta \in [-\pi, -\pi] \)

Hence

\[ \int_{C_{-}} z^{1/2} \, dz = \int_{-\pi}^{-\pi/2} e^{i\theta/2} (e^{i\theta}) \, d\theta = \int_{-\pi}^{-\pi/2} e^{3i\theta/2} \, d\theta = \left. i e^{-3i\theta/2} \right|_{-\pi}^{-\pi/2} \]

\[ \int_{C_{+}} z^{1/2} \, dz = \int_{\pi/2}^{\pi} e^{i\theta/2} (e^{i\theta}) \, d\theta = \int_{\pi/2}^{\pi} e^{3i\theta/2} \, d\theta = \left. i e^{3i\theta/2} \right|_{\pi/2}^{\pi} \]

Adding

\[ \int_{C} z^{1/2} \, dz = \int_{\pi/2}^{\pi} \left( e^{3i\theta/2} + e^{-3i\theta/2} \right) \, d\theta = 2i \int_{\pi/2}^{\pi} \cos(\theta) d\theta = \frac{4}{3} i \sin \left( \frac{3\theta}{2} \right) \bigg|_{\pi/2}^{\pi} = \frac{4}{3} \left( -1 - \frac{\sqrt{2}}{2} \right) \]