Upper bounds for Contour Integrals

Lemma: Let $w(t)$ be continuous on $[a,b]$

$$\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt$$

Pf wlog integrals are non zero so $\exists\, r_0, \theta_0$ s.t

$$r_0 e^{i\theta_0} = \int_a^b w(t)\,dt$$

$$r_0 = \int_a^b e^{-i\theta_0} w(t)\,dt \qquad r_0 \text{ real!}$$

$$r_0 = \int_a^b \mathrm{Re}\left(e^{-i\theta_0} w(t)\right)dt$$

$\mathrm{Re}\, z \leq |z|$

$$r_0 \leq \int_a^b |e^{-i\theta_0} w(t)|\,dt$$

$$r_0 \leq \int_a^b |w(t)|\,dt$$

Take absolute value of both sides for result $\square$

Theorem Let contour C have length L and suppose $f(z)$ is continuous on C and $\exists M$ s.t.

$$(1) \qquad |f(z)| \le M \qquad \forall z \in C$$

Then

$$\left| \int_C f(z)\,dz \right| \le ML$$

Pf
$$\left| \int_C f(z)\,dz \right| = \left| \int_a^b f(z(t))\,z'(t)\,dt \right| \le \int_a^b |f(z)\,z'|\,dt$$

Given the assumption (1)

$$(2) \qquad \left| \int_C f(z)\,dz \right| \le M \int_a^b |z'|\,dt$$

arclength!

The integral on the right is arclength $z(t) = x(t) + iy(t)$

$$\int_a^b |z'|\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = L$$

Hence (2) $\Rightarrow$

$$\left| \int_C f(z)\,dz \right| \le ML$$

Triangle inequalities

(1) $$|z_1 + z_2| \le |z_1| + |z_2|$$

(2) $$|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$$

Pf of (2)

(3) $$|z_1| = |z_1 + z_2 - z_2| \le |z_1 + z_2| + |z_2|$$

Thus, from (3) and repeating for $|z_2|$

$$\left.\begin{aligned} |z_1 + z_2| &\geq |z_1| - |z_2| \\ |z_1 + z_2| &\geq |z_2| - |z_1| \end{aligned}\right\} \text{implies (2)}$$

EXAMPLE Polynomial upper bound. Let $z = Re^{i\theta}$

$$\begin{aligned} |3z^2 + 2z - 7| &\le 3|z|^2 + 2|z| + 7 \\ &\le 3R^2 + 2R + 7 \end{aligned}$$

EXAMPLE Reciprocal Polynomial upper bound

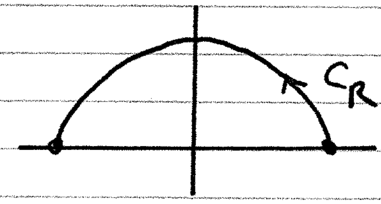
$z = Re^{i\theta}$
$$\left.\begin{aligned} |z^2 - 3z + 1| &\geq |z^2 - 3z| - 1 \\ &\geq |z|^2 - 3|z| - 1 \\ &\geq R^2 - 3R - 1 \end{aligned}\right\} \text{R big enough so positive}$$

Thus for sufficienty large R

$$\frac{1}{|z^2 - 3z + 1|} \le \frac{1}{R^2 - 3R - 1}$$

EXAMPLE Upper bound for the integral

$$I_n = \int_{C_R} \frac{dz}{z^n}$$



$z = Re^{i\theta}$

$\theta \in [0, \pi]$

Done by evaluating

$$I_n = \int_0^{\pi} R^{-n} e^{-in\theta} (Rie^{i\theta})\, d\theta$$

$$I_n = \int_0^{\pi} R^{1-n} \cdot i e^{i(1-n)\theta}\, d\theta$$

$$|I_n| \leq R^{1-n} \int_0^{\pi} |ie^{i(1-n)\theta}|\, d\theta$$

$$|I_n| \leq \pi R^{1-n} \qquad n = 2, 3, 4, \ldots$$

In particular,

$$|I_n| \to 0 \quad \text{as } R \to \infty \quad \text{if } n \geq 2.$$

For $n = 1$ one can quickly show

$$I_1 = \pi i \qquad \forall R.$$

<u>EXAMPLE</u> Upper bound $\displaystyle\int_{|z|=3} \frac{z+1}{z^3-1}\,dz = \int_{|z|=3} f(z)\,dz$

Length L easy: $L = 2\pi R = 6\pi$.

Upper bound on integrand $f(z)$ in two parts

(1) $|z+1| \leq |z| + 1 = 3+1 = 4$

For denominator

$$|z^3-1| \geq |z|^3 - 1 = 3^3 - 1 = 26$$

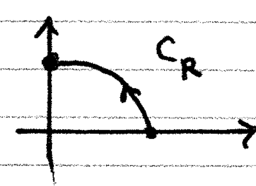
Conclude that for $|z| = 3$

$$\left|\frac{z+1}{z^3-1}\right| \leq \frac{4}{26} \equiv M$$

By the integral upper bound theorem

$$\left|\int_C \frac{z+1}{z^3-1}\,dz\right| \leq ML = \frac{4}{26}\cdot(6\pi) = \frac{12}{13}\pi$$

<u>EXAMPLE</u> $\displaystyle\int_{C_R} \frac{z^{1/2}}{z^2-1}\,dz$ on



$z = Re^{i\theta}$

$\theta \in [0, \frac{\pi}{2}]$

Similarly

$$\left|\int_{C_R} \frac{z^{1/2}}{z^2-1}\,dz\right| \leq \frac{R^{1/2}}{R^2-1}\cdot\left(\frac{\pi}{2}R\right) \to 0 \quad \text{as } R \to \infty$$