Theorem Let $f(z)$ be continuous (analytic) on an open connected set D. The following are equivalent

(a) $f(z)$ has an antiderivate $F(z)$ such that $F'(z) = f(z)$ on D

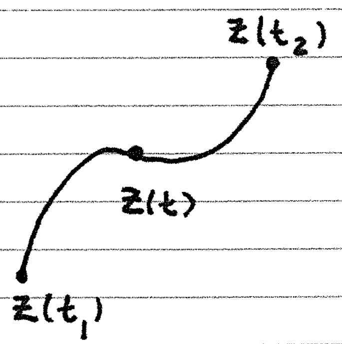
(b) Let C be any contour lying in D with endpoints z_1 and z_2. Then

$$\left.\int_C f(z)\,dz = F(z_2) - F(z_1)\right\}\text{Fundamental Theorem of Calculus}$$

(c) For all closed curves C in D

$$\oint_C f(z)\,dz = 0$$

Pf (a) $\Rightarrow$ (b).



$$\int_C f(z)\,dz = \int_{t_1}^{t_2} f(z(t))\,z'(t)\,dt$$

chain rule

$$= \int_{t_1}^{t_2} \frac{dF}{dt}\,dt$$

$\Uparrow$

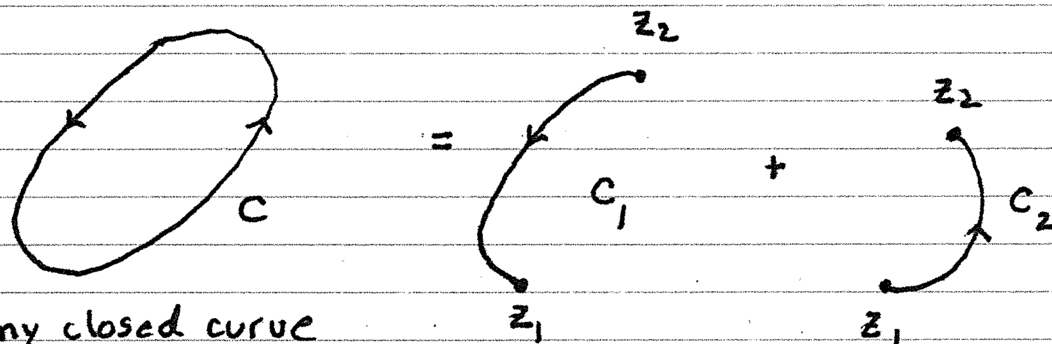
for any curve connecting z_1 and z_2

$$= F(z(t_2)) - F(z(t_1))$$

$$= F(z_2) - F(z_1)$$

(the answer is the same)

Pf (b) $\Rightarrow$ (c)



any closed curve

By (b)

$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

$$= [F(\cancel{z_1}) - F(\cancel{z_2})] + [F(\cancel{z_2}) - F(\cancel{z_1})]$$

$$= 0$$

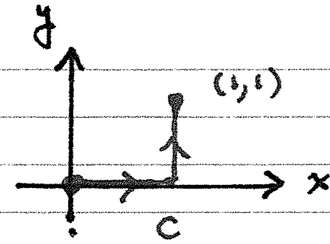
Pf (c) $\Rightarrow$ (a) Sketch

$$F(z) \equiv \int_a^z f(s)\,ds$$

Take limit as $\Delta z \to 0$ of

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \underbrace{\frac{1}{\Delta z}\int_z^{z+\Delta z} [f(s) - f(z)]\,ds}_{\downarrow\; 0}$$

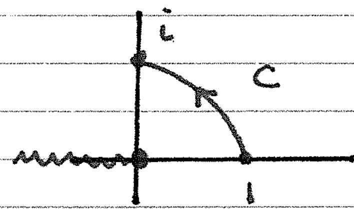
EXAMPLE $\displaystyle\int_C z^2\,dz$



Regular calculus

$$\int_C z^2\,dz = \tfrac{1}{3}z^3\Big|_{(0,0)}^{(1,1)} = \tfrac{2}{3}(-1+i)$$

EXAMPLE $\displaystyle\int_C \mathrm{Log}\,z\,dz$



$|z|=1$

By conventional IBP methods one can show

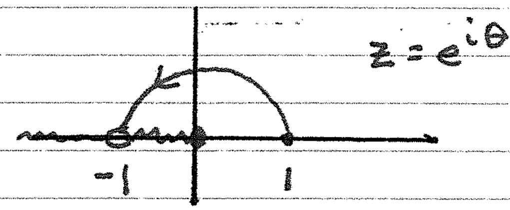
$$F(z) = z\,\mathrm{Log}\,z - z$$

is the antiderivative of integrand $f(z) = \mathrm{Log}\,z$

$$\begin{aligned}\int_C \mathrm{Log}\,z\,dz &= F(i) - F(1)\\ &= [i\,\mathrm{Log}\,i - i] - [1\,\cancel{\mathrm{Log}\,1} - 1]\\ &= (1 - \tfrac{\pi}{2}) - i\end{aligned}$$

EXAMPLE Near a cut is fine.

$$I = \int_C z^i \, dz$$



Use the principal branch z^i defined by

$$(1) \qquad f(z) \equiv z^i \equiv \exp(i \operatorname{Log} z)$$

Noting

$$\begin{aligned}
\frac{d}{dz} z^{i+1} &= \exp((i+1)\operatorname{Log} z) \cdot \frac{d}{dz} \operatorname{Log} z \\
&= \exp((i+1)\operatorname{Log} z) \cdot \frac{1}{z} (1+i) \\
&= \exp((1+i)\operatorname{Log} z) \exp(-\operatorname{Log} z)(1+i) \\
&= z^i \cdot (1+i)
\end{aligned}$$

We have (formally) the antiderivative $F(z)$ of $f(z)$ in (1):

$$F(z) = \frac{1}{(1+i)} z^{i+1}$$

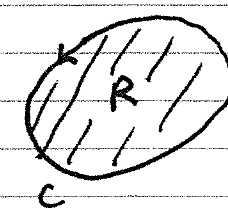
Thus

$$\begin{aligned}
\int_C z^i \, dz &= \frac{1}{(1+i)} (-1)^{i+1} - \frac{1}{(1+i)} \cancelto{1}{(1)^{i+1}} \\
&= \frac{1}{(1+i)} \left\{ \exp(i\pi(1+i)) - 1 \right\} \\
&= \frac{1}{(i+1)} \left(1 - e^{-\pi} \right)
\end{aligned}$$

Green's Theorem in the Plane.

Let C be a simple closed curve (positive) bounding region $R \subset \mathbb{R}^2$. Let P, Q and their derivatives be continuous on and interior to C. Then

$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$$



Cauchy - Goursat Theorem

Let $f(z)$ be analytic on and interior to a simple closed curve C. Then

$$\oint_C f(z)\,dz = 0$$

Proof for $f'(z)$ continuous. Recall

$$\oint_C f(z)\,dz = \left(\int_C u\,dx - v\,dy\right) + i\left(\int v\,dx + u\,dy\right)$$

$\downarrow$ Green's Thm $\downarrow$

$$= \iint_R (-v_x - u_y)\,dA + i \iint_R (u_x - v_y)\,dA$$

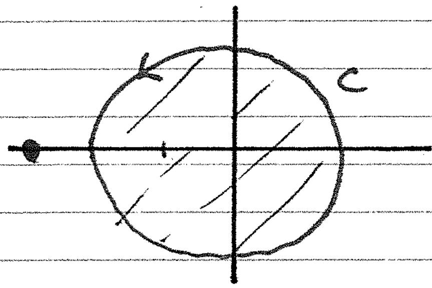
$\uparrow$ $\uparrow$

both vanish by CR-eqns

$$u_x = v_y$$
$$u_y = -v_x$$

Remark: Holds if $f'(z)$ cont. dropped. LONG PROOF in text.

EXAMPLE $f(z) = \dfrac{z^2}{z+3}$ $|z|=2$ positive



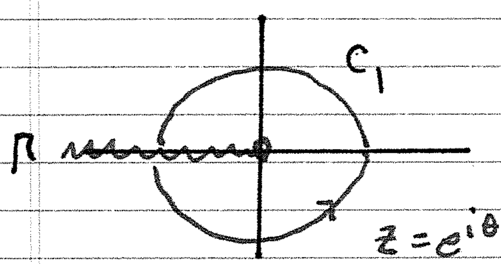
f fails to be analytic at one point "singularity"

$z = -3$

Hypotheses true $\Rightarrow \oint_C f(z)\,dz = 0$

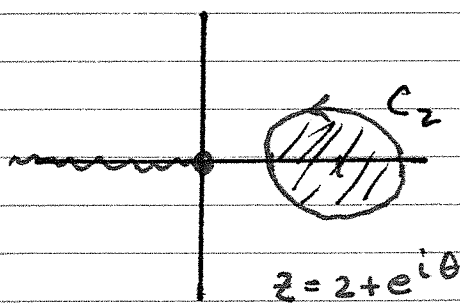
EXAMPLE $f(z) = \text{Log}\, z$ Principal Branch

Analytic everywhere except on cut Γ



Hypotheses fail.

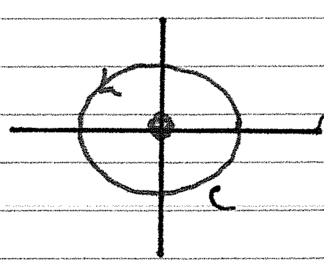
Conclude nothing



Hypotheses satisfied

$$\oint_{C_2} \text{Log}\, z\, dz = 0$$

EXAMPLE $f(z) = \dfrac{1}{z}$ on unit circle $z = e^{it}$



$f(z)$ not analytic at $z=0$ singularity hence Thm does not apply! We previously calculated

$$\oint_C \frac{1}{z}\, dz = 2\pi i \neq 0$$