Singularities

Defn: $z_0$ is a singularity of $f(z)$ if $f(z)$ is not analytic at $z_0$ (but is analytic at some point in every neighbourhood $N_r(z_0)$).

Defn: A singular point $z_0$ of $f(z)$ is isolated if $\exists \epsilon > 0$ s.t. $z_0$ is the only singular point in $|z - z_0| < \epsilon$.

EXAMPLES

$$f(z) = \frac{1}{z(z^2 + 9)} \quad z = 0, \pm 3i$$ isolated singularities

$$f(z) = \sqrt{z}$$ all points on branch cut are sing. pts.

EXAMPLES $x$ isolated sing pt $\quad \text{mmm cut}$

$$N_r(z_0) \quad xxx \quad N_r(z_0)$$

not isolated isolated

Defn: $z_0$ is a pole of order $m$ if $\exists g(z)$ analytic s.t.

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

EXAMPLE $z_0 = 0$ is a pole of order 3 for

$$f(z) = \frac{e^z}{z^3}$$
Multiply Connected Domains

Cauchy-Goursat tells us the (zero) value of closed curve integrals when \( f(z) \) is analytic inside and on \( C \).

What if \( C \) contains a finite number of isolated singularities?

\[ \text{C}^+ \text{ top} \]

\[ \text{C}^- \text{ bottom} \]

Let \( f(z) \) be analytic inside shaded region.

\[ C = C^+ \cup C^- \quad \text{full outside loop.} \]

Given integrals on \( \pm L_k \) cancel. Cauchy-Goursat \( \Rightarrow \)

\[
\oint_{C} f(z) \, dz = \sum_{k=1}^{n} \oint_{C_k} f(z) \, dz
\]

Line integral around \( C \) is the sum of the integrals around the \( k=1, \ldots, n \) singularities.
**Example**

Compute \( \oint_C \frac{1}{z} \, dz \) where \( C \) is any curve surrounding the origin.

Using Theorem (1) there exists \( \varepsilon \):

\[ \oint_C f(z) \, dz = \oint_{C_\varepsilon} \frac{1}{z} \, dz \]

where \( z = \varepsilon e^{i\theta}, \theta \in [0, 2\pi) \). Previously have computed this integral:

\[ \oint_C f(z) \, dz = \oint_{\partial B(0,\varepsilon)} \frac{1}{z} \, dz = 2\pi i \]

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**Example**

Does:

\[ \oint_C \frac{z+2}{\sin(z/2)} \, dz = \oint_{C_1} \frac{z+2}{\sin(z/2)} \, dz \]

Hinges on where singularities of \( f(z) \) are. Defined by

\[ \sin(z/2) = 0 \quad z = 0, 2\pi n \quad n = \pm 1, \pm 2 \]

Since \( z = 0 \) sole sing. pt interior to \( C \) (and \( C_1 \)) the integrals are the same.
Example: Evaluate \( \int_{C} \frac{z}{z^2 + 1} \, dz \)

Closed semicircle \( |z| = 2 \)

\( x \) are sing pts of \( f(z) \)

By Cauchy-Goursat for \( C_\varepsilon \) defined by \( |z - i| = \varepsilon \)

\[
I = \oint_{C} \frac{z}{z^2 + 1} \, dz = \oint_{C_\varepsilon} \frac{z}{z^2 + 1} \, dz
\]

Next we compute a partial fraction expansion

\[
f(z) = \frac{z}{z^2 + 1} = \frac{A}{z+i} + \frac{B}{z-i} = \frac{(A+B)z + i(B-A)}{z^2 + 1}
\]

We deduce \( A = B = \frac{1}{2} \) so

\[
f(z) = \frac{\frac{1}{2}}{z+i} + \frac{\frac{1}{2}}{z-i}
\]

Thus (1) above becomes

\[
I = \frac{1}{2} \oint_{C} \frac{1}{z+i} \, dz + \frac{1}{2} \oint_{C_\varepsilon} \frac{1}{z-i} \, dz
\]

\( 0 \) since integral is analytic

\( 2\pi i \) by direct calculation

\[
I = 0 + \pi i
\]

\[
I = \pi i
\]
Cauchy Integral Formula

Let $f(z)$ be analytic inside and on simple closed curve $C$. For every interior point $z_0$,

$$(1) \quad f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} \, dz$$

Proof (abbreviated)  
Let $C_p$ be circular (small) curve. By Cauchy-Goursat multi connect,

$$\oint_C g(z) \, dz = \oint_{C_p} g(z) \, dz$$
then

$$\oint_C \frac{f(z)}{z-z_0} \, dz - f(z_0) \oint_C \frac{dz}{z-z_0} = \oint_{C_p} \frac{f(z)-f(z_0)}{z-z_0} \, dz$$

$2\pi i \to 0$ by conty from which (1) follows.

Since RHS $\to 0$ the terms on left must equal $\Rightarrow (1)$ above.

Why does RHS $\to 0$ ?? $f(z)$ continuous at $z_0$ $\Rightarrow$

$$|f(z)-f(z_0)| < \varepsilon \quad \text{if} \quad |z-z_0| < \delta$$

Then

$$\left| \oint_{C_p} \frac{f(z)-f(z_0)}{z-z_0} \, dz \right| < \frac{\varepsilon}{\rho} (2\pi \rho) = 2\pi \varepsilon \quad (\varepsilon \text{ arbitrary})$$

so long as $\rho < \delta$. 
**Example**

Let $C$ be the unit circle counterclockwise. Evaluate

$$ I = \oint_C \frac{\cos z}{z(z^2+9)} \, dz $$

First locate singular points

$z = 0, \pm 3i$

and define $f(z)$ in the following

$$ I = \oint_C \frac{\cos z}{z(z^2+9)} \, dz = \oint_C \frac{\cos z}{z} - \oint_C \frac{\cos z}{z^2+9} \, dz $$

Written this way we can apply the Cauchy Integral formula.

$$ I = 2\pi i \cdot f(0) $$

$$ I = 2\pi i \left( \frac{1}{9} \right) $$

$$ I = \frac{2\pi i}{9} $$
EXAMPLE  Evaluate

\[ I = \oint_C \frac{z^3}{z^2 + 1} \, dz \]

\[ |z-i| = 1 \]

Rewrite integrand

\[ I = \oint_C \frac{f(z)}{(z-i)} \, dz = \oint_C \frac{1}{(z-i)} \frac{z^3}{(z+i)} \, dz \]

By Cauchy's Integral formula

\[ I = 2\pi i f(i) \quad z_0 = i \]

\[ I = \pi i \left( \frac{i^3}{2i} \right) \]

\[ I = -\pi i \]
Cauchy Integral Formulas

Suppose $f(z)$ is analytic inside and on $C$. Then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}} ds$$

For a proof of (1) see the text.

Remark An interesting conclusion from (1) is that if $f(z)$ is analytic then it has derivatives of all orders.

$$f'(z) \text{ exists } \Rightarrow f^{(n)}(z) \text{ exists } \forall n=1,2,\ldots$$

Here we write out all of our integral formulae

$$0 = \frac{1}{2\pi i} \oint_C f(z) \, dz \quad \text{Cauchy-Goursat}$$

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z)} \, ds$$

2) Take derivatives in $z$

$$f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z)^2} \, ds$$

$$f''(z) = \frac{2!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^3} \, ds$$
**Pf**: Outline of \( n = 1 \) proof

Given
\[
f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s-z} \, ds
\]

we form the difference
\[
f(z + \Delta z) - f(z) = \frac{1}{2\pi i} \oint_C f(s) \left[ \frac{1}{(s-z-\Delta z)} - \frac{1}{(s-z)} \right] ds
\]

algebraically combine

From which
\[
\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z-\Delta z)(s-z)} \, ds
\]

\[
\begin{align*}
\frac{f(z + \Delta z) - f(z)}{\Delta z} &= \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z-\Delta z)(s-z)} \, ds \\
&\downarrow \\
\frac{f'(z)}{2\pi i} \oint_C \frac{f(s)}{(s-z)^2} \, ds
\end{align*}
\]

For \( n > 1 \) do similar limits as in
\[
f'(z + \Delta z) - f'(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s-z} \left[ \frac{1}{(s-z-\Delta z)^2} - \frac{1}{(s-z)^2} \right] ds
\]
**Example** Let $C$ be the unit circle $z = e^{i\theta}$

\[
\oint_{C} \frac{e^{z}}{z} \, dz = \oint_{C} \frac{e^{z}}{z-0} \, dz = 2\pi i \cdot e^{0} = 2\pi i
\]

**Example** Pole order 3 on $|z| = 1$

\[
\oint_{|z|=1} \frac{e^{3z}}{z^{3}} \, dz = \frac{2\pi i}{2!} \cdot \frac{f^{(2)}(0)}{2!} = \frac{9\pi i}{2!} = \frac{9\pi i}{2} \cdot \frac{e^{3z}}{z^{3}}
\]

**Example** $C$ is square drawn

\[
I = \oint_{C} \frac{\sin z}{z^{2}+1} \, dz
\]

Since $z_0 = i$ is the sole singularity inside $C$

\[
I = \oint_{C} \frac{\sin z}{(z+i) \cdot (z-i)} \, dz = \oint_{C} \frac{f(z)}{z-i} \, dz
\]

Hence

\[
I = (\pi i) \cdot f(i) = (\pi i) \cdot \frac{\sin i}{z} = \pi \cdot \sin i
\]
Example: Multiple poles

\[ I = \oint_C \frac{z}{z^2 + 1} \, dz \]

By Cauchy’s integral theorem, we have

\[ I = \oint_{C_1} \frac{z}{z^2 + 1} \, dz + \oint_{C_2} \frac{z}{z^2 + 1} \, dz \]

\[ I = \oint_{C_1} \frac{z}{(z+i)(z-i)} \, dz + \oint_{C_2} \frac{z}{(z+i)(z-i)} \, dz \]

\[ I = 2\pi i f_1(i) + 2\pi i f_2(-i) \]

\[ I = \pi i + \pi i \]

\[ I = 2\pi i \]
\[ Ex \quad I = \oint_C \frac{dz}{(z^2 + 4)^2} \quad \text{where } C \text{ is the contour } |z-2i| = 1 \]

**Integrand factors**

\[ \frac{1}{(z+2i)^2(z-2i)^2} = \frac{f(z)}{(z-2i)^2} \]

where \( f(z) = (z+2i)^{-2} \).

By Cauchy's Integral Formula

\[ I = \frac{2\pi i}{1} f'(2i) \]

Compute the derivative \( f'(z) = -2(z+2i)^{-3} \)

\[ I = 2\pi i \left( -2 \right) (2i+2i)^{-3} \]

\[ I = -4\pi i \cdot \frac{1}{(4i)^3} \]

\[ I = \frac{-4\pi i}{-4^3 i} \]

\[ I = \frac{\pi}{16} \]