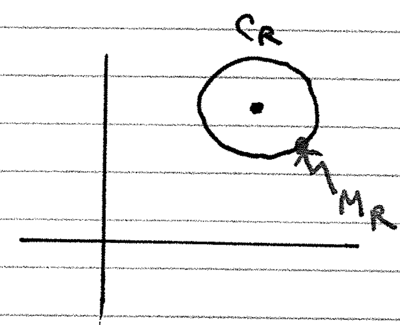
Cauchy's inequality

Let C_R be a circle of radius R centered at z_0. If $f(z)$ is analytic inside and on C_R then

$$|f^{(n)}(z_0)| \le \frac{n!\, M_R}{R^n}$$



where

$$M_R = \max_{z \in C_R} |f(z)|$$

Proof Follows from Cauchy Integral formula and ML-inequality:

$$f^{(n)}(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

Let $z = z_0 + Re^{i\theta}$

$$f^{(n)}(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z)\, R i \, d\theta}{R^{n+1}}$$

$$|f^{(n)}(z_0)| = \frac{1}{2\pi} \cdot \frac{1}{R^n} \left| \int_0^{2\pi} f(z)\, i \, d\theta \right|$$

$$\le \frac{1}{2\pi} \cdot \frac{1}{R^n} \cdot 2\pi M_R$$

Liouville's Theorem

If $f(z)$ is analytic and bounded on $\mathbb{C}$ then it is constant

Proof First note by supposition $\exists M$ s.t.

$$|f(z)| \leq M \qquad \forall z \in \mathbb{C}$$

Use Cauchy inequality with $n=1$.

Let C_R be circle radius R center z_0.

depends on R

$$(1) \qquad |f'(z_0)| \leq \frac{M_R}{R} \qquad M_R = \max_{C_R} |f(z)|$$

But, since $M_R \leq M$

constent

$$(2) \qquad |f'(z_0)| \leq \frac{M}{R}$$

True $\forall R$. In particular as $R \to \infty$ we conclude

$$f'(z_0) = 0$$

where z_0 arbitrary. Conclude

$$f(z) = \text{constant}.$$

Fundamental Theorem of Algebra

Any polynomial

$$P(z) = a_0 + a_1 z + \cdots a_n z^n \qquad a_n \neq 0$$

with $n \geq 1$, has at least one root.

Proof By contradiction. Suppose $P(z) \neq 0 \ \forall z \in \mathbb{C}$. Then

$$f(z) = \frac{1}{P(z)}$$

is entire. But one can show*

$$|f(z)| = \left|\frac{1}{P(z)}\right| < \frac{2}{|a_n| R^n} \qquad |z| > R$$

So $f(z)$ is bounded outside C_R. It is inside too hence

$$f(z) \text{ bounded on } \mathbb{C}$$

By Liouville's Theorem f entire and bounded on $\mathbb{C} \Rightarrow f(z)$ constant

This is a contradiction.

Conclude $\exists z_0$ s.t. $P(z_0) = 0$.

Corollary: $P(z) = a_n(z-z_1)(z-z_2)\cdots(z-z_n)$

where $z_K \in \mathbb{C}$

* sketch on next page.

Bounds on $P_n(z)$ Text pg 13

(1) $$P(z) = (a_n + \underset{\uparrow}{w})\, z^n = \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots \frac{a_{n-1}}{z} \right) z^n$$

Next bounds for wz^n

$$w z^n = a_0 + a_1 z + \cdots a_{n-1} z^{n-1}$$

$$|w| \le \underbrace{\frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots \frac{|a_{n-1}|}{|z|}}$$

each quotient $\le \frac{|a_n|}{2n}$ for $|z| > R$ and R big

$$|w| \le n \frac{|a_n|}{2n} = \frac{1}{2} |a_n| \qquad |z| > R$$

Hence

$$|a_n + w| \ge \big| |a_n| - |w| \big| > \frac{|a_n|}{2}$$

Apply to (1)

$$|P(z)| \ge \frac{|a_n|}{2} |z^n| > \frac{|a_n| R^n}{2}$$

$$\frac{1}{P(z)} \le \frac{2}{|a_n| R^n}$$

Gauss Mean Value Theorem

(1) $$C_\rho \equiv \{ z : z = z_0 + \rho e^{i\theta} \}$$

If $f(z)$ is analytic in and on C_ρ

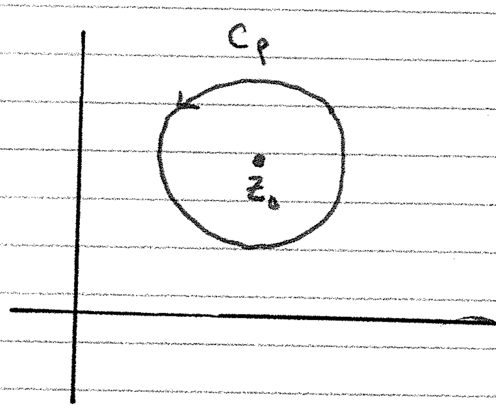
(2) $$f(z_0) = \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(z)}{z - z_0}\, dz$$

Given (1), (2) becomes

$$f(z_0) = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + \rho e^{i\theta})\, \cancel{i}\cancel{\rho e^{i\theta}}\, d\theta}{\cancel{\rho e^{i\theta}}}$$

Simplifying

$$\boxed{f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \overbrace{f(z_0 + \rho e^{i\theta})}^{\text{"average"}}\, d\theta}$$



value at senter $f(z_0)$ is the average on the bounding circle.