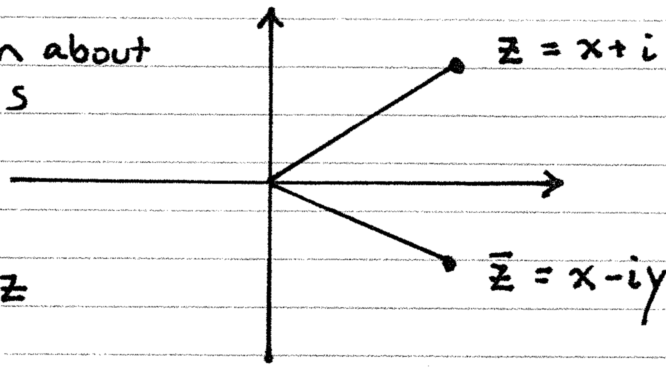
Modulus, Conjugate and Geometry

Defn The complex conjugate $\bar{z}$ of z is

$$\bar{z} = x - iy$$

reflection about $y = 0$ axis



Note: $\bar{\bar{z}} = z$

Theorem Let $z, z_1, z_2 \in \mathbb{C}$.

(i) $|z| = \sqrt{z\bar{z}}$

(ii) $|z| = |\bar{z}|$

(iii) $|z_1 z_2| = |z_1||z_2|$

(iv) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(v) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(vi) $|z_1 + z_2| \leq |z_1| + |z_2|$

(vii) $z^{-1} = \dfrac{\bar{z}}{|z|^2}$

Most of these are easy to prove.

Proof of (i)

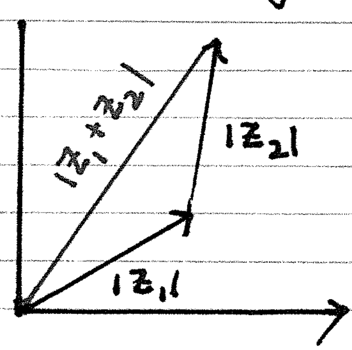
$$z\bar{z} = (x+iy)(x-iy) = x^2+y^2 = |z|^2$$

Proof of (iii)

$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$$\begin{aligned} |z_1 z_2|^2 &= (x_1x_2-y_1y_2)^2 + (x_1y_2+x_2y_1)^2 \\ &= x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2 \\ &= (x_1^2+y_1^2)(x_2^2+y_2^2) \\ &= |z_1|^2 \cdot |z_2|^2 \end{aligned}$$

Proof of (vi) Triangle inequality



Follows from the fack the sum of the lengths of two sides of a triangle is larger than the length of the remaining side

Proof of (vii)

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

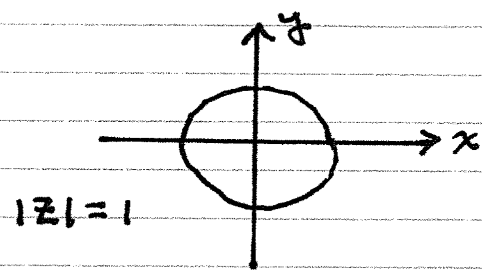
and clearly $z \cdot z^{-1} = 1$.

Sets defined using modulii

EXAMPLE Set of z such that

(1) $$|z| = 1$$

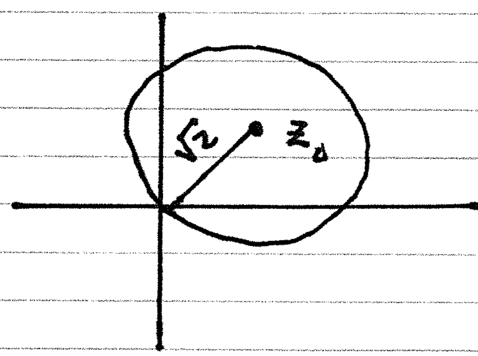
is equivalent to $x^2+y^2=1$ hence (1) describes a unit circle



EXAMPLE Set of z such that

(2) $$|z-z_0| = \sqrt{2} \qquad z_0 = 1+i$$

All z that are a distance $\sqrt{2}$ from z_0



circle center z_0
radius $\sqrt{2}$

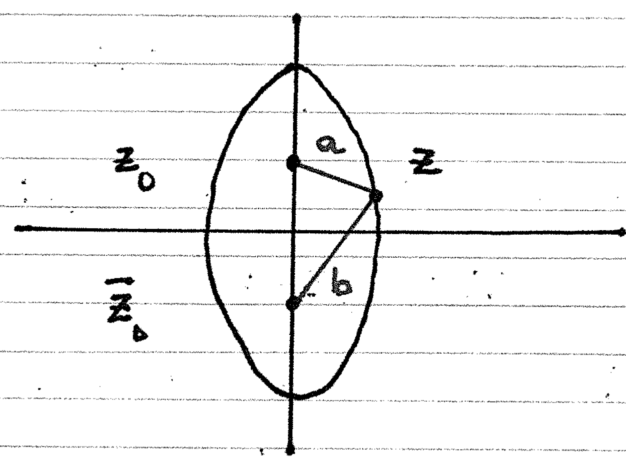
Note: $|z_0| = \sqrt{2}$ so that circle goes thru $(0,0)$.

EXAMPLE Set of z such that

(3) $|z - z_0| + |z - \bar{z}_0| = 20$

where $z_0 = 5i$.

In words, the sum of the distance from z to z_0 and z to $\bar{z}_0$ is constant 20. This is the defn of an ellipse with focii z_0 and $\bar{z}_0$.

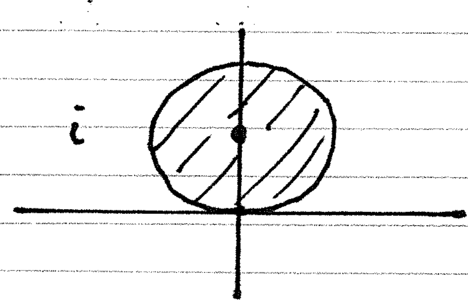


$a + b = 20$

EXAMPLE Set of z s.t. $\frac{1}{|2z-1|} = 2 \iff$

$|z - \frac{1}{2}| = \frac{1}{4}$ circle center $(\frac{1}{2}, 0)$ radius $\frac{1}{2}$

EXAMPLE Set of z s.t. $|z - i| \leq 1$



interior of circle centered at $z_0 = i$ and radius $r = 1$.

Bounds using moduli and triangle inequality

A simple generalization of $|z_1 + z_2| \leq |z_1| + |z_2|$ is

$$|z_1 + \cdots z_n| \leq |z_1| + |z_2| + \cdots |z_n|$$

EXAMPLE Find an upper bound to $P(z) = 2 + z + z^2$ for all z on circle $|z| = 3$

$$|P(z)| = |2 + z + z^2|$$

$$\leq |2| + |z| + |z^2|$$

$$\leq |2| + |z| + |z|^2$$

$$\leq 2 + 3 + 3^2$$

$$\leq 14$$

Complex Polynomials

$$P(z) = a_0 + a_1 z + \cdots a_n z^n$$

where $z \in \mathbb{C}$ and $a_k \in \mathbb{C}$ are complex coefficients.

Later we need to place bounds on how large

$$|P(z)| \qquad \frac{1}{|P(z)|}$$

are. This will occur in integrals.

EXAMPLE Find an upper bound to $|P(z)|$ on $|z| = R$ for $R > 1$

$$|P(z)| = |a_0 + a_1 z + \cdots a_n z^n|$$

$$\leq |a_0| + |a_1||z| + \cdots |a_n||z|^n$$

Now let

$$M = \max_k |a_k|$$

so that

$$|P(z)| \leq M(1 + R + R^2 + \cdots R^n)$$

$$|P(z)| \leq (n+1) M R^n$$

The result

$$\boxed{|P(z)| \leq (n+1) M R^n}$$

proves $|P(z)| \to \infty$ as $|z| \to \infty$ like "real" polynomials

EXAMPLE In the text

$$\frac{1}{|P(z)|} \leq \frac{2}{|a_n| R^n}$$