Modulus, Conjugate and Geometry

Defn. The complex conjugate $\bar{z}$ of $z$ is

$$\bar{z} = x - iy$$

reflection about $y = 0$ axis

Note: $\bar{\bar{z}} = z$

Theorem. Let $z, z_1, z_2 \in \mathbb{C}$.

(i) $|z| = \sqrt{z\bar{z}}$

(ii) $|z| = |\bar{z}|$

(iii) $|z_1z_2| = |z_1||z_2|$

(iv) $\frac{z_1 + z_2}{\bar{z}_1 + \bar{z}_2}$

(v) $\frac{z_1}{z_2} = \frac{\bar{z}_1}{\bar{z}_2}$

(vi) $|z_1 + z_2| \leq |z_1| + |z_2|$

(vii) $z^{-1} = \frac{\bar{z}}{|z|^2}$

Most of these are easy to prove.
Proof of (i)

\[ z \overline{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2 \]

Proof of (iii)

\[
|z_1z_2|^2 = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2
\]
\[= x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2\]
\[= (x_1^2 + y_1^2)(x_2^2 + y_2^2)\]
\[= |z_1|^2 \cdot |z_2|^2\]

Proof of (vi) Triangle inequality

Follows from the fact that the sum of the lengths of two sides of a triangle is larger than the length of the remaining side.

Proof of (vii)

\[ z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2} \]

and clearly \( z \cdot z^{-1} = 1 \).
Sets defined using modulii.

**EXAMPLE** Set of \( z \) such that

(1) \[ |z| = 1 \]

is equivalent to \( x^2 + y^2 = 1 \) hence (1) describes a unit circle.

**EXAMPLE** Set of \( z \) such that

(2) \[ |z - z_0| = \sqrt{2} \] \[ z_0 = 1 + i \]

All \( z \) that are a distance \( \sqrt{2} \) from \( z_0 \).

Note: \( |z_0| = \sqrt{2} \) so that circle goes thru \((0,0)\).
**Example** Set of $z$ such that

$$|z - z_0| + |z - ar{z}_0| = 20$$

where $z_0 = 5i$.

In words, the sum of the distance from $z$ to $z_0$ and $z$ to $\bar{z}_0$ is constant 20. This is the definition of an ellipse with foci $z_0$ and $\bar{z}_0$.

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**Example** Set of $z$ s.t. $\frac{1}{12z - 11} = 2$.

$$|z - \frac{1}{2}| = \frac{1}{4} \quad \text{circle center } \left(\frac{1}{2}, 0\right)$$

radius $\frac{1}{2}$

**Example** Set of $z$ s.t. $|z - i| \leq 1$

interior of circle centered at $z_0 = i$

and radius $r = 1$. 
Bounds using moduli and triangle inequality

A simple generalization of $|z_1 + z_2| \leq |z_1| + |z_2|$ is

$$|z_1 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

**Example** Find an upper bound to $P(z) = 2 + z + z^2$

for all $z$ on circle $|z| = 3$

$$|P(z)| = |2 + z + z^2|$$

$$\leq |2| + |z| + |z^2|$$

$$\leq |2| + |z| + |z|^2$$

$$\leq 2 + 3 + 3^2$$

$$\leq 14$$

Complex Polynomials

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$

where $z \in \mathbb{C}$ and $a_k \in \mathbb{C}$ are complex coefficients.

Later we need to place bounds on how large

$$\frac{1}{|P(z)|}$$

are. This will occur in integrals.
**EXAMPLE** Find an upper bound to $|P(z)|$ on $|z| = R$ for $R > 1$

$$|P(z)| = |a_0 + a_1 z + \cdots + a_n z^n|$$

$$\leq |a_0| + |a_1||z| + \cdots + |a_n||z|^n$$

Now let

$$M = \max_k |a_k|$$

so that

$$|P(z)| \leq M \left(1 + R + R^2 + \cdots + R^n\right)$$

$$|P(z)| \leq (n+1) M R^n$$

The result

$$|P(z)| \leq (n+1) M R^n$$

proves $|P(z)| \to \infty$ as $|z| \to \infty$ like "real" polynomials

**EXAMPLE** In the text

$$\frac{1}{|P(z)|} \leq \frac{2}{|a_n| R^n}$$