Euler's Formula

$$\boxed{e^{ix} = \cos x + i \sin x \qquad x \in \mathbb{R}}$$

Pf (presumes knowledge of Taylor Series)

$$e^{ix} = 1 + (ix) + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \cdots$$

$$e^{ix} = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right)$$

$$e^{ix} = \cos x + i \sin x \qquad \square$$

A remarkable formula can be found by setting $x = \pi$

$$e^{i\pi} = -1$$

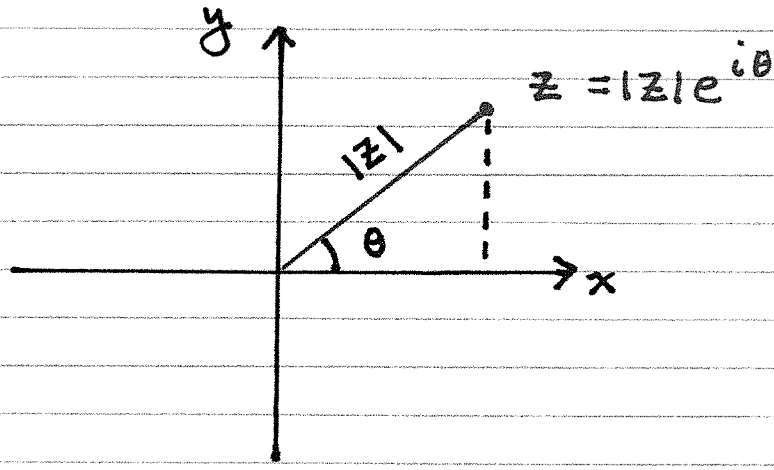
or

$$\boxed{e^{i\pi} + 1 = 0}$$

which relates all the most important numbers

$$0 \quad 1 \quad e \quad \pi \quad i$$

<u>Polar Representation</u>



Given Euler's formula

(1) $$z = |z|e^{i\theta} = |z|(\cos\theta + i\sin\theta)$$

where $\theta = \arg(z)$. Hence

$$\mathrm{Re}\, z = |z|\cos\theta$$

$$\mathrm{Im}\, z = |z|\sin\theta$$

It is worth noting $|e^{i\theta}| = 1$ in

$$\boxed{z = |z|e^{i\theta}}$$

$$\boxed{\begin{array}{l}\underline{\text{Theorem}}\\ \qquad z_1 z_2 = |z_1 z_2|\, e^{i(\theta_1+\theta_2)}\end{array}}$$

Pf is direct but lengthy

(1) $z_1 z_2 = |z_1||z_2|\,(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

Take real and imaginary parts

$$\mathrm{Re}(z_1 z_2) = |z_1||z_2|\,(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$$
$$= |z_1||z_2|\,\cos(\theta_1+\theta_2)$$

and

$$\mathrm{Im}(z_1 z_2) = |z_1||z_2|\,(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= |z_1||z_2|\,\sin(\theta_1+\theta_2)$$

Hence (1) becomes

$$z_1 z_2 = |z_1||z_2|\,\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)$$
$$z_1 z_2 = |z_1||z_2|\,e^{i(\theta_1+\theta_2)} \qquad //$$

EXAMPLE $z_1 = i$ and $z_2 = 1+i$

Using Arg (Principal) we have

$$z_1 = e^{i\pi/2} \qquad \text{Arg}(z_1) = \frac{\pi}{2}$$

$$z_2 = \sqrt{2}\, e^{i\pi/4} \qquad \text{Arg}(z_2) = \frac{\pi}{4}$$

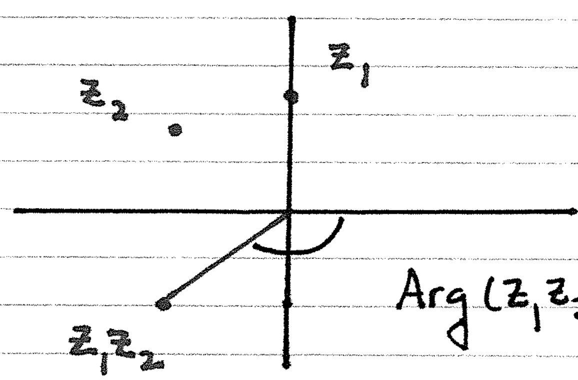
$$z_1 z_2 = \sqrt{2}\, e^{i\frac{3\pi}{4}}$$

Here $\text{Arg}(z_1 z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2$

EXAMPLE $z_1 = i$ and $z_2 = -1+i$

$$z_1 z_2 = \sqrt{2}\, e^{i\left(\frac{\pi}{2} + \frac{3\pi}{4}\right)}$$

$$z_1 z_2 = \sqrt{2}\, e^{i\frac{5\pi}{4}} = -1 - i$$



$\text{Arg}(z_1 z_2) \neq \frac{5\pi}{4}$

Here

$$\text{Arg}(z_1 z_2) \neq \text{Arg}\, z_1 + \text{Arg}\, z_2$$

Theorem DeMoivre's Formula

Let $r = |z|$. For any integer $n \geq 1$

$$z^n = r^n(\cos(n\theta) + i\sin n\theta)$$

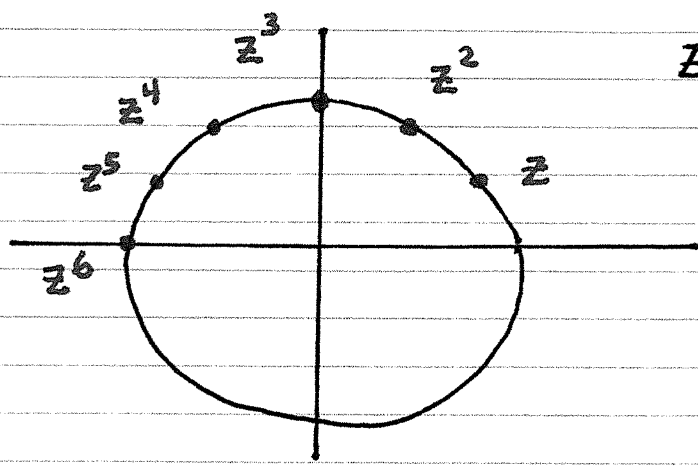
Pf: Induction with $z = z_1 = z_2$.

EXAMPLE Powers of $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$.

In polar we have

$$z = e^{i\theta} \qquad \theta = \pi/6$$

noting $|z| = 1$.



$$z^n = e^{in\theta}$$

If we use $\theta = \arg(z) \in [0, 2\pi)$ then

$$\arg z^7 = \frac{7\pi}{6}$$

but if we use $\theta = \operatorname{Arg}(z) \in (-\pi, \pi]$

$$\operatorname{Arg} z^7 = -\frac{5\pi}{6}$$

Inverse of z in polar

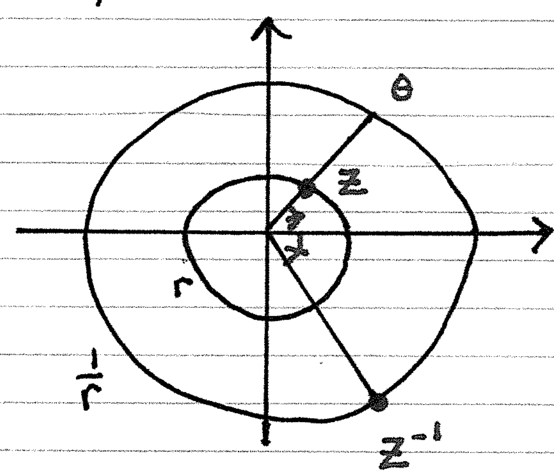
Again $r \equiv |z|$. Then $z = re^{i\theta}$.
It is easy to verify

$$\boxed{z^{-1} = \frac{1}{r} e^{-i\theta}}$$

since the product

$$z z^{-1} = r \cdot \frac{1}{r} e^{i(\theta-\theta)} = e^{0i} = 1$$

Graphically



Circles of radius r, r^{-1}

EXAMPLE $z = 1+i$ has polar form $z = \sqrt{2}\, e^{i\theta/4}$

Can easily find

$$z^{-1} = \frac{1}{\sqrt{2}} e^{-i\theta/4} = \frac{\bar{z}}{|z|^2} = \frac{1}{2}(1-i)$$

Arguments and Products / Powers

From the polar representation of $z_1 z_2$ we can easily deduce

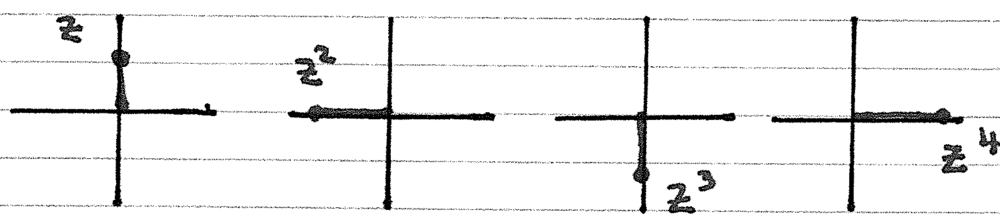
$$|z_1 z_2| = |z_1||z_2|$$

$$\text{Arg}(z_1 z_2) = \text{Arg}\, z_1 + \text{Arg}\, z_2 \mod 2\pi$$

If z_k aren't both in quadrant I it may be the case $\text{Arg}(z_1 z_2)$ and $\text{Arg}\, z_1 + \text{Arg}\, z_2$ are off by a multiple of 2π.

Illustrate issue by an example

<u>EXAMPLE</u> $z = i$ and $z = e^{i\theta}$ where $\theta = \pi/2$ using $\text{Arg}(z)$



$$\text{Arg}\, z = \frac{\pi}{2}$$

$$\text{Arg}\, z^2 = \pi$$

$$\text{Arg}\, z^3 = -\pi/2 \neq \text{Arg}\, z + \text{Arg}\, z + \text{Arg}\, z = \frac{3\pi}{2}$$

$$\text{Arg}\, z^4 = 0$$

EXAMPLE Compute $(1+i)^7$

Of course we wouldn't expand out $(1+i)^7$. That would be too much work. Instead note that in polar this is the same as:

$$z^7 \qquad z = \sqrt{2}\, e^{i\pi/4}$$

Hence

$$z^7 = (\sqrt{2})^7 e^{i\,7\pi/4}$$

$$z^7 = 8\sqrt{2}\, e^{i(2\pi - \pi/4)} \quad \text{change to Arg}$$

$$z^7 = 8\sqrt{2}\, e^{-i\pi/4}$$

$$z^7 = 8\sqrt{2}\,(1-i)$$

EXAMPLE Use de Moivre's formula to derive identities for $\cos(2\theta)$ and $\sin(2\theta)$

For $z = e^{i\theta} = \cos\theta + i\sin\theta$, deMoivre's formula $\Rightarrow$

$$z^2 = (\cos 2\theta + i\sin 2\theta)$$

$$= (\cos\theta + i\sin\theta)^2$$

$$= (\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos\theta$$

Equating real and imaginary parts

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$