Square root (Nonpolar form)

For any $z = x + iy$, by $w = \sqrt{z}$ we mean a solution of

(1) $\qquad w^2 = z \qquad\qquad w = a + ib$

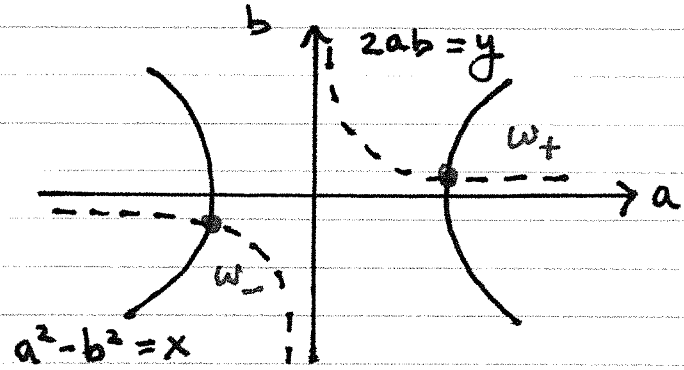
Equating real and imaginary parts in (1):

(2) $\qquad a^2 - b^2 = x$

(3) $\qquad 2ab = y$

Must solve (2)-(3) for (a, b). For fixed (x, y) eqns (2)-(3) describe hyperbolae in the (a, b) plane. We will restrict our attention to the case $x > 0$, $y > 0$.

Fig 1



Two roots.

Since $|w|^2 = |z| = \sqrt{x^2+y^2}$ we know $a^2 + b^2 = \sqrt{x^2+y^2}$. This and (2) then imply

(4) $\qquad a^2 = \frac{1}{2}(x + \sqrt{x^2+y^2})$

(5) $\qquad b^2 = \frac{1}{2}(-x + \sqrt{x^2+y^2})$

Then we have two roots with the signs of a, b depending on Fig 1

Thus we have (for $x, y > 0$)

$$\sqrt{x+iy} = \pm(\alpha + i\beta)$$

where

$$(6) \quad \alpha = \sqrt{\frac{x+\sqrt{x^2+y^2}}{2}} \qquad \beta = \sqrt{\frac{-x+\sqrt{x^2+y^2}}{2}}$$

<u>EXAMPLE</u> Find all solutions of

$$\omega^2 = 2+3i = z$$

Here $x, y > 0$ and $|z| = \sqrt{13}$. Using (6)

$$\alpha = \sqrt{\frac{2+\sqrt{13}}{2}} = \mathrm{Re}(\omega)$$

$$\beta = \sqrt{\frac{-2+\sqrt{13}}{2}} = \mathrm{Im}(\omega)$$

Hence

$$\sqrt{2+3i} = \pm(\alpha + i\beta)$$

N-th roots (Polar)

Given De Moivre's formula, it is easy to find the n roots z of

(1) $$z^n = w = re^{i\theta} \qquad n \geq 2 \text{ integer}$$

Informally $z = \sqrt[n]{w}$ where

(2) $$z_k = r^{1/n}(\cos \psi_k + i \sin \psi_k)$$

(3) $$\psi_k = \frac{\theta}{n} + \frac{2\pi k}{n} \qquad k = 0, 1, \ldots (n-1)$$

Pf: Using exponential notation

$$z_k = r^{1/n} e^{i\psi_k}$$

$$z_k^n = r \; e^{in\psi_k}$$

$$z_k^n = r e^{i\theta} \cdot \underbrace{e^{2\pi k i}}_{\text{one for } k = \text{integer}}$$

$$z_k^n = re^{i\theta} \qquad \square$$

Remark: All roots have the same magnitude, i.e. $|z_k| = r^{1/n}$ and thus all lie on a circle of radius $r^{1/n}$. They also are all separated by the same angle

$$\Delta\psi = \frac{2\pi}{n}$$

EX Find all roots of $z^3 = 8$ using $\theta = \arg(z) \in [0, 2\pi)$

Here $w = 8 = 8e^{i\theta}$ where $|w| = 8$, $\theta = 0$.

$$z_k = r^{1/3}(\cos\psi_k + i\sin\psi_k) \qquad r = |w| = 8$$

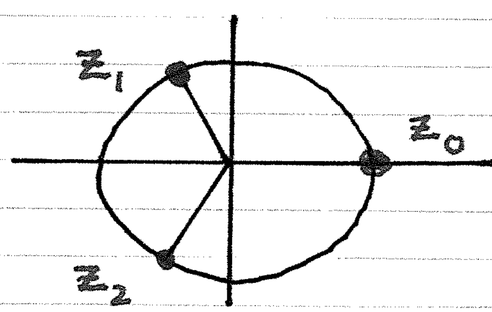
$$\psi_k = \frac{\theta}{n} + \frac{2\pi k}{n} = 0, \frac{2\pi}{3}, \frac{4\pi}{3} \qquad (k = 0, 1, 2)$$

Explicitly

$$z_0 = 2(\cos 0 + i \sin 0) = 2$$

$$z_1 = 2\left(\cos\left(\tfrac{2\pi}{3}\right) + i\sin\left(\tfrac{2\pi}{3}\right)\right) = -1 + \sqrt{3}\, i$$

$$z_2 = 2\left(\cos\left(\tfrac{4\pi}{3}\right) + i\sin\left(\tfrac{4\pi}{3}\right)\right) = -1 - \sqrt{3}\, i$$



All roots on $|z| = 2$

Separated by

$$\Delta\psi = \frac{2\pi}{3}$$

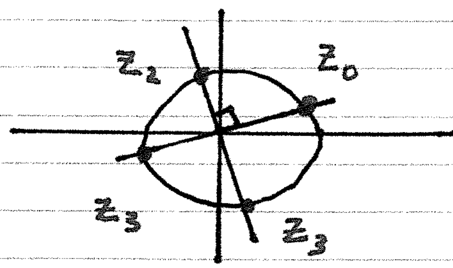
EXAMPLE $\quad z^4 = 16(1+i) = 16\sqrt{2}\, e^{i\theta}$

Here $r = 16\sqrt{2}$, $\theta = \arg(w) = \pi/4$ and $n = 4$.

$$z_k = \alpha\,(\cos\psi_k + i\sin\psi_k)$$

where $\alpha = 2^{9/8}$ and the angles

$$\psi_k = \frac{1}{4}\left(\frac{\pi}{4} + 2\pi k\right) = \frac{\pi}{16}, \frac{9\pi}{16}, \frac{17\pi}{16}, \frac{25\pi}{16}$$

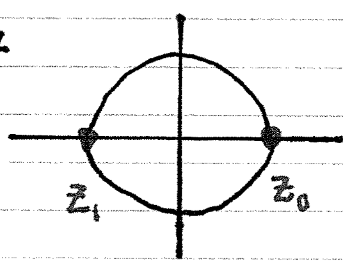


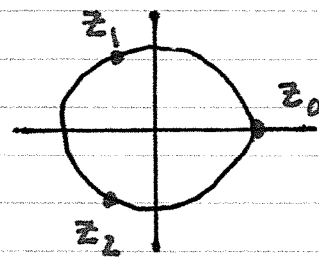
$\Delta\psi = \dfrac{\pi}{2}$

EXAMPLE Roots of unity : graphical

$$z^n = 1$$

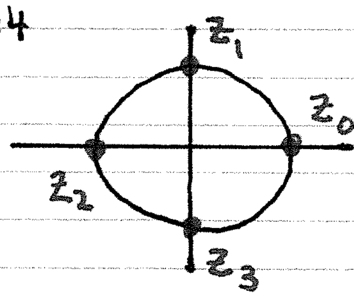
$n=2$

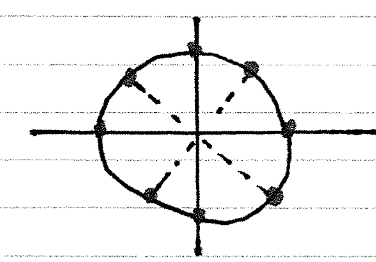




$n=3$

$n=4$





$n=8$

EXAMPLE $z \equiv e^{i\pi/8}$. Use nonpolar form of square root to find an expression for $\cos(\pi/8)$

(1) $$\omega = z^2 = e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i) = x+iy$$

We need $\mathrm{Re}(\sqrt{\omega})$. Reference theory:

$$\alpha = \sqrt{\frac{x+\sqrt{x^2+y^2}}{2}} = \cdots = \frac{1}{2}\sqrt{2+\sqrt{2}}$$

$$\beta = \sqrt{\frac{x-\sqrt{x^2+y^2}}{2}} = \cdots = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

Thus, $\sqrt{\omega} = \alpha + i\beta$ and

$$z = e^{i\pi/8} = \underbrace{\frac{1}{2}\sqrt{2+\sqrt{2}}}_{\cos(\pi/8)} + \underbrace{\frac{1}{2}\sqrt{2-\sqrt{2}}}_{\sin(\pi/8)}\, i$$

$$\boxed{\cos\left(\frac{\pi}{8}\right) = \frac{1}{2}\sqrt{2+\sqrt{2}}}$$

Quadratic Formula

For any $a, b, c \in \mathbb{C}$ it is still true the roots of

$$P(z) = az^2 + bz + c = 0$$

are given by

$$\boxed{z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

EXAMPLE Factor $P(z) = z^2 + bz + c$ where $b = -3+i$ and $c = 2-i$

If roots of $P(z)$ are $z_\pm$ then $P(z) = (z - z_+)(z - z_-)$

$$z_\pm = \frac{1}{2}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

(1) $$z_\pm = \frac{1}{2}\left(-3 + i \pm \sqrt{\underbrace{-2i}_{w}}\right)$$

In polar $w = 2e^{-\pi/2}$ hence $\sqrt{w} = \sqrt{2}\, e^{-i\pi/4}$.

(2) $$\sqrt{w} = \sqrt{2}\left(\cos\frac{\pi}{4} - i \sin\frac{\pi}{4}\right) = 1 - i$$

Using (2) in (1) we get

$$z_+ = 1$$

$$z_- = -2 + i$$

to conclude

$$P(z) = (z-1)(z+2-i)$$