Points

Definition 0.1 *z_0 is an <u>interior point</u> of S if $\exists N_r(z_0) \subset S$.*

Definition 0.2 *z_0 is an <u>exterior point</u> of S if $\exists N_r(z_0)$ containing no point of S.*

Definition 0.3 *z_0 is an <u>boundary point</u> of S if it is neither an interior nor exterior point.*

Definition 0.4 *z_0 is a <u>limit point</u> of S if every $N_r(z_0)$ contains a point $z \neq z_0$ in S.*

Note that if z_0 is a limit point, there is a sequence $\{z_n\}$ contained in S such that $z_n \to z_0$. Limit points are sometimes called accumulation points because of this property.

Sets in $\mathbb{C}$

Definition 0.5 *A set S is* **open** *if every point $z \in S$ is an interior point.*

Definition 0.6 *A set S is* **closed** *if every limit point of S is a point of S.*

Definition 0.7 *If S' is the set of limit points of S then $\bar{S} \equiv S \cup S'$ is the closure of S.*

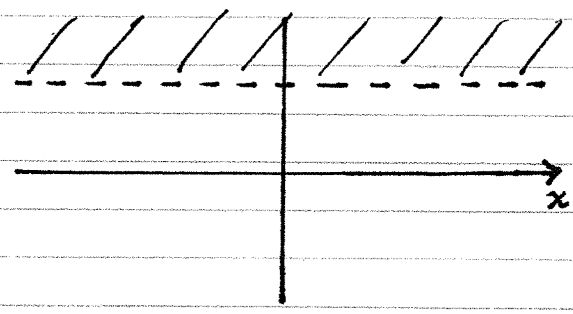
Definition 0.8 *A set S is* **path-connected** *if for every $z_1, z_2 \in S$ there is a polygonal line Γ that is contained entirely in S and connectins z_1 and z_2.*

Definition 0.9 *A (complex)* **domain** *is any nonempty open set S*

Definition 0.10 *A set S is* **bounded** *if there exists a constant $M > 0$ such that*

$$|z| < M \qquad , \qquad \forall z \in S$$

EXAMPLE $\operatorname{Im}(z) > 1$

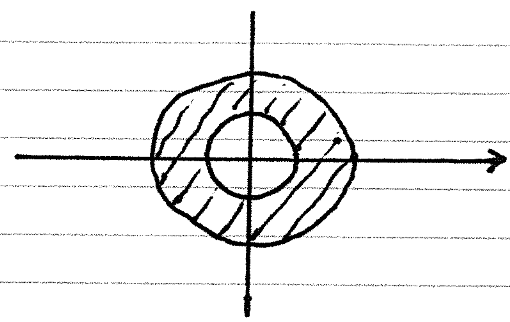


open

unbounded

connected

EXAMPLE $1 \le |z| \le 2$



closed

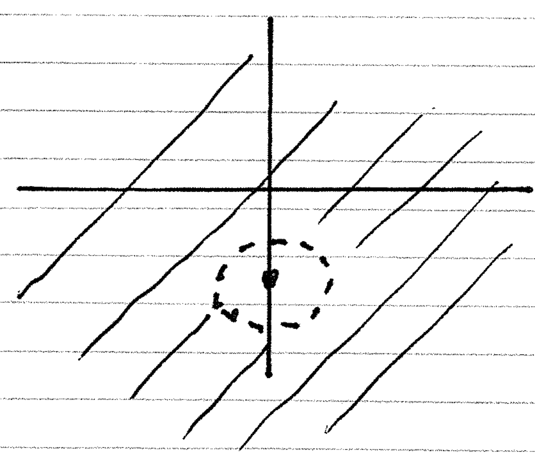
bounded

connected

EXAMPLE $\dfrac{1}{|z+i|} < 4$

is the same as

$|z+i| > \frac{1}{4}$ center $(0,-i)$ radius $\frac{1}{4}$

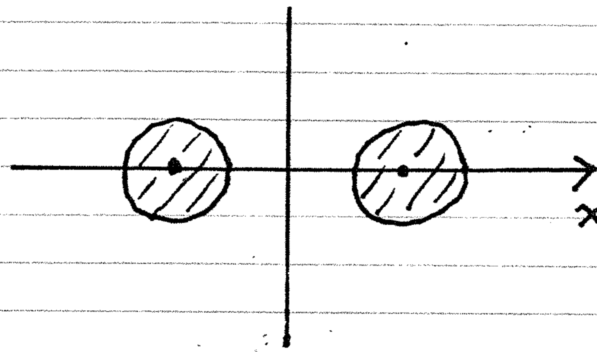


open

unbounded

connected

EXAMPLE $|z+1| \le \frac{1}{2}$ or $|z-1| \le \frac{1}{2}$

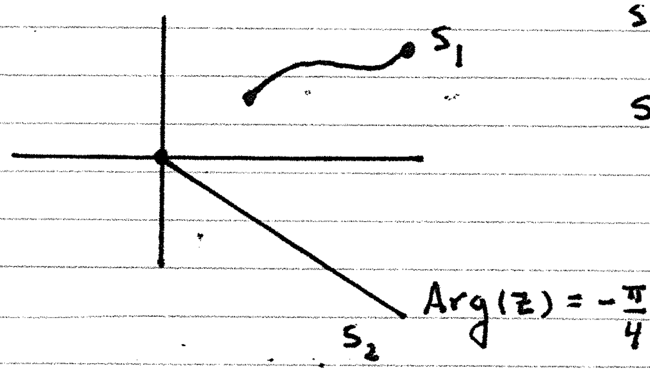


closed

bounded

not connected

EXAMPLE Curves in $\mathbb{C}$

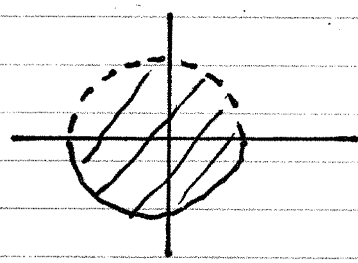


S_1: closed, bounded

S_2: closed, unbounded

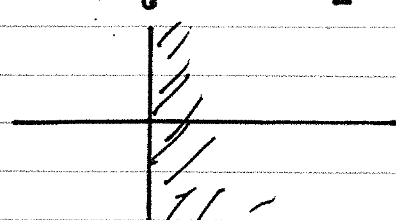
EXAMPLE $\mathbb{C}$ is both open and closed, unbounded

EXAMPLE



neither open nor closed

EXAMPLE $|\text{Arg}(z)| \le \frac{\pi}{2}$ closed, unbounded



$\text{Re}\, z \ge 0$

Functions of a Complex variable

Let S be the domain of

$$f : S \to \mathbb{C}$$

The range R_f is the set

$$R_f = \{ w \in \mathbb{C} : \exists z \in S,\ f(z) = w \}$$

As a mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$ there exist functions u and v such that

$$f(z) = u(x,y) + i\,v(x,y) \qquad z = x + iy$$

$$f(z) = u(r,\theta) + i\,v(r,\theta) \qquad z = re^{i\theta}$$

EXAMPLE $\quad f(z) = z^2$

$$f(z) = (x^2 - y^2) + i\,2xy$$

$$f(z) = r^2 \cos 2\theta + i\,r^2 \sin 2\theta$$

EXAMPLE $\quad$ Square root $f(z) \equiv \sqrt{z}$ where we choose the branch having $\operatorname{Re}\sqrt{z} > 0$.

In polar with $\theta \equiv \operatorname{Arg} z$

$$f(z) = \sqrt{z}$$

$$f(z) = r^{1/2} \cos\left(\frac{\theta}{2}\right) + i\,r^{1/2} \sin\left(\frac{\theta}{2}\right)$$

EXAMPLE $$f(z) = \frac{z}{z+\bar{z}}$$

The domain of f excludes the $x=0$ axis, and

$$f(z) = \frac{x+iy}{(x+iy)+(x-iy)} = \underset{u(x,y)}{\underset{\uparrow}{\frac{1}{2}}} + \frac{1}{2}\, i\, \underset{v(x,y)}{\underset{\uparrow}{\frac{y}{x}}}$$

EXAMPLE $$f(z) = |z|^2 = (x^2+y^2) + 0 \cdot i$$

EXAMPLE $$f(z) = \operatorname{Arg} z = \theta$$

EXAMPLE Rational function

$$f(z) \equiv \frac{z+1}{z-1} \qquad z = x+iy$$

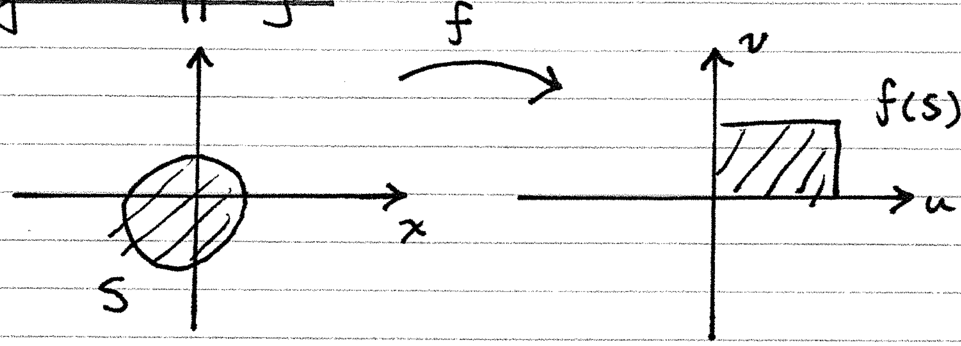
then

$$f(z) = \frac{(z+1)(\bar{z}-1)}{|z-1|^2} = \frac{[(x+1)+iy][(x-1)-iy]}{(x-1)^2+y^2}$$

Expand and collect real/imaginary parts

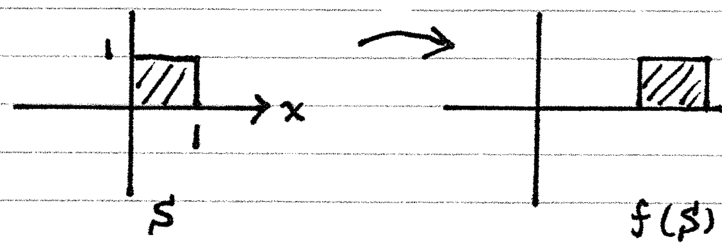
$$f(z) = \underbrace{\frac{x^2+y^2-1}{(x-1)^2+y^2}}_{u(x,y)} + i\, \underbrace{\frac{-2y}{(x-1)^2+y^2}}_{v(x,y)}$$

Region Mappings



Generally, f maps regions into regions

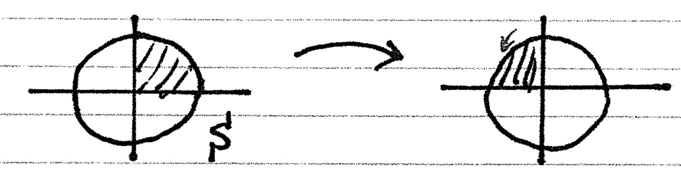
EXAMPLE $f(z) = z + 2$



translation

EXAMPLE $f(z) = e^{i\pi/2} z$

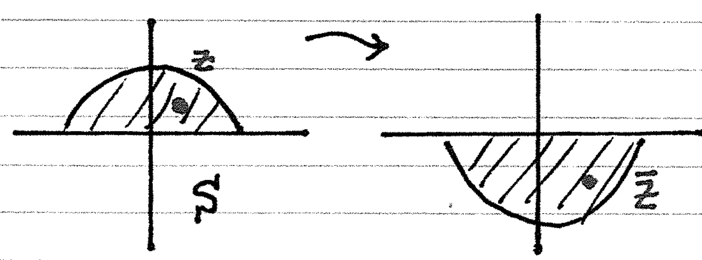
$|z| = 1$

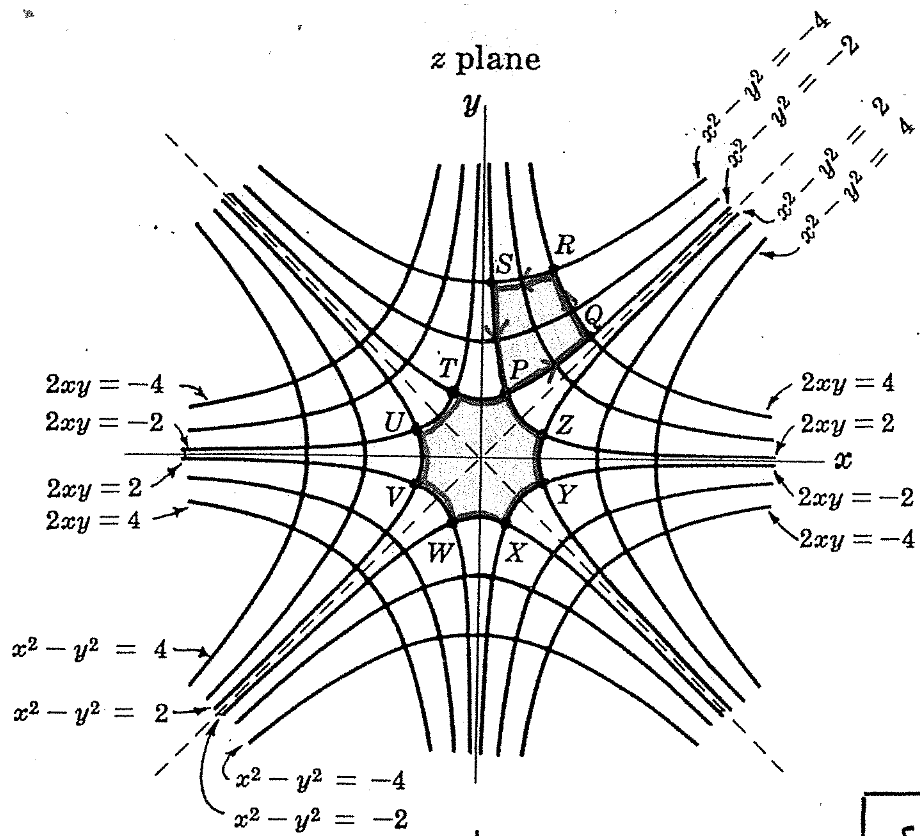


rotation by $\pi/2$

EXAMPLE $f(z) = \bar{z}$

reflection

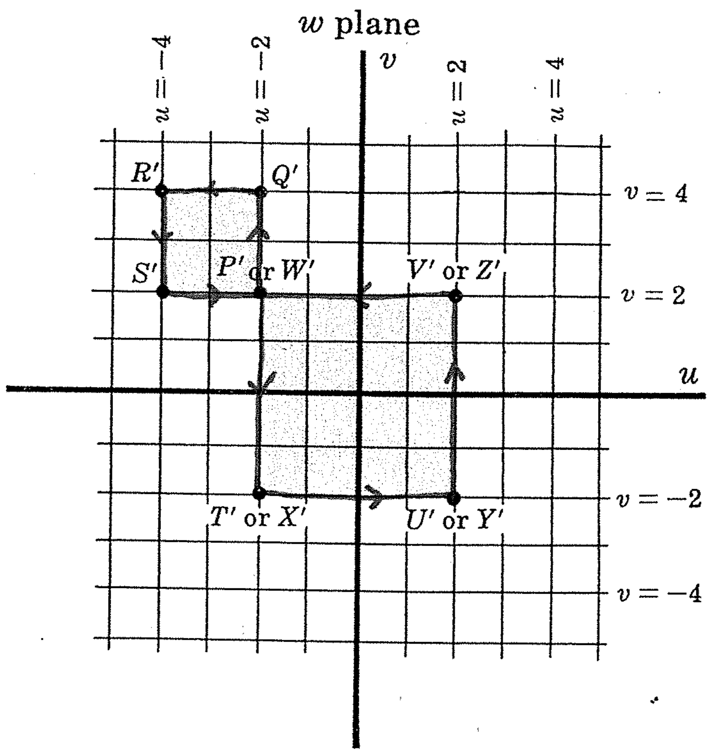




$f(z) = z^2$

$\text{Re}\, z^2 = x^2 - y^2$

$\text{Im}\, z^2 = 2xy$



Summary of limits

Definition 0.1 *We say the* **limit**

$$\lim_{z \to z_0} f(z) = L$$

if every $\epsilon > 0$ *there is a* $\delta > 0$ *such that*

$$|z - z_0| < \delta \quad \Rightarrow \quad |f(z) - L| < \epsilon$$

$f(z_0)$ *need not be defined at* z_0

Definition 0.2 *We say* $f(z)$ *is* **continuous** *at* z_0 *if*

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Here, $f(z_0)$ *is defined at* z_0

All of the main limit properties that apply to real valued functions also apply to complex valued functions. Below is a partial list where it is assumed all the limits exist.

$$\lim_{z \to z_0} af(z) = a \lim_{z \to z_0} f(z) \qquad a \in \mathbb{C} \tag{0.1}$$
$$\lim_{z \to z_0} (f(z) + g(z)) = \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) \tag{0.2}$$
$$\lim_{z \to z_0} f(z)g(z) = \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z) \tag{0.3}$$
$$\lim_{z \to z_0} f(g(z)) = f(g(z_0)) \tag{0.4}$$

Theorem 0.3 *For* $f : \mathbb{C} \to \mathbb{C}$,

$$f(z) = u(x,y) + iv(x,y)$$

for some functions u, v. *For* $z_0 = (x_0, y_0)$, *the function* $f(z)$ *is continuous if and only if both* $u(x,y)$ *and* $v(x,y)$ *are continuous at* (x_0, y_0).

EXAMPLE Prove $\lim_{z \to z_0} z^2 = z_0^2$

Choose $\varepsilon > 0$ and define

$$M = \max_{|z - z_0| \le r} |z + z_0|$$

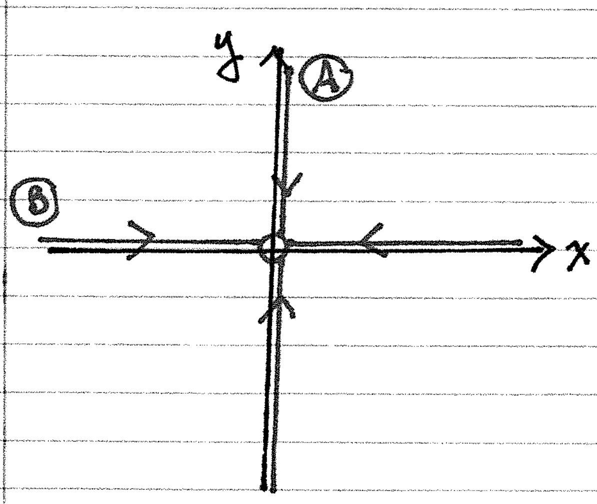
max exists. Then if $\delta = \frac{\varepsilon}{2M}$ we have

$$|z^2 - z_0^2| = |z + z_0||z - z_0| \le M|z - z_0| < \varepsilon/2$$

if $|z - z_0| < \delta$.

EXAMPLE Show $f(z) = \frac{\bar{z}}{z}$ not cont. at $z = 0$.

$$f(z) = \frac{x - iy}{x + iy}$$

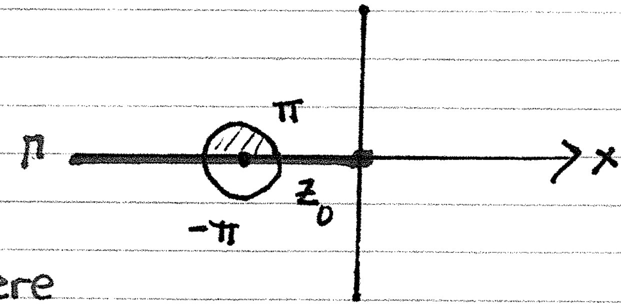


$$\lim_{y \to 0,\, x = 0} f(z) = -1 \quad \text{(A)}$$

$$\lim_{x \to 0,\, y = 0} f(z) = +1 \quad \text{(B)}$$

Since (A) $\neq$ (B), $\lim_{z \to 0} f(z)$ DNE and f not cont at 0.

EXAMPLE $f(z) = \text{Arg}\, z$ discont'y $x \le 0, y = 0$



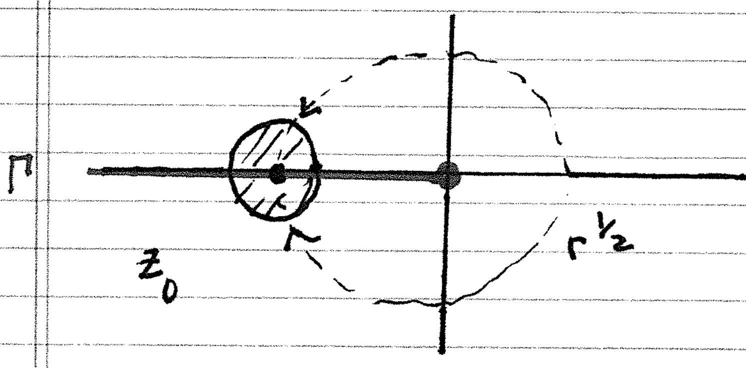
discont everywhere on cut Γ

Shows $N_r(z_0)$ where z_0 is on Γ

EXAMPLE Principal Branch $f(z) = \sqrt{z}$

$z = re^{i\theta}$ $\quad f(z) \equiv r^{1/2} e^{i\theta/2}$ $\quad \theta \in (-\pi, \pi]$

is also discontinuous on $x \le 0, y = 0$



$\lim\limits_{\theta \to \pi} f(z) = i r^{1/2}$

$\lim\limits_{\theta \to -\pi^+} f(z) = -i r^{1/2}$

attains values $\pm i r^{1/2}$ for every $N_r(z_0)$. Not cont.

EXAMPLE $\lim\limits_{z \to 2} \bar{z} = 2$

(1) $|\bar{z} - 2| = |\overline{\bar{z} - 2}| = |z - 2| < \varepsilon$

if $\delta = \varepsilon$, i.e. if $|z-2| < \delta = \varepsilon$ then (1) true