Points

Definition 0.1 \( z_0 \) is an interior point of \( S \) if \( \exists N_r(z_0) \subseteq S \).

Definition 0.2 \( z_0 \) is an exterior point of \( S \) if \( \exists N_r(z_0) \) containing no point of \( S \).

Definition 0.3 \( z_0 \) is a boundary point of \( S \) if it is neither an interior nor exterior point.

Definition 0.4 \( z_0 \) is a limit point of \( S \) if every \( N_r(z_0) \) contains a point \( z \neq z_0 \) in \( S \).

Note that if \( z_0 \) is a limit point, there is a sequence \( \{z_n\} \) contained in \( S \) such that \( z_n \to z_0 \). Limit points are sometimes called accumulation points because of this property.

Sets in \( \mathbb{C} \)

Definition 0.5 A set \( S \) is open if every point \( z \in S \) is an interior point.

Definition 0.6 A set \( S \) is closed if every limit point of \( S \) is a point of \( S \).

Definition 0.7 If \( S' \) is the set of limit points of \( S \) then \( \bar{S} \equiv S \cup S' \) is the closure of \( S \).

Definition 0.8 A set \( S \) is path-connected if for every \( z_1, z_2 \in S \) there is a polygonal line \( \Gamma \) that is contained entirely in \( S \) and connects \( z_1 \) and \( z_2 \).

Definition 0.9 A (complex) domain is any nonempty open set \( S \)

Definition 0.10 A set \( S \) is bounded if there exists a constant \( M > 0 \) such that 

\[ |z| < M \quad , \quad \forall z \in S \]
**Example** \[ \text{Im}(z) > 1 \]

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**Example** \[ 1 \leq |z| \leq 2 \]

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**Example** \[ \frac{1}{1+|z+i|} < 4 \]

is the same as \[ |z+i| > \frac{1}{4} \]

**center** \((0, -i)\) **radius** \(\frac{1}{4}\)

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(open, unbounded, connected)

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EXAMPLE $1Z + 11 \leq \frac{1}{2}$ or $1Z - 11 \leq \frac{1}{2}$

\begin{center}
\includegraphics[width=0.5\textwidth]{example1.png}
\end{center}

closed bounded not connected

EXAMPLE Curves in $\mathbb{C}$

\begin{center}
\includegraphics[width=0.5\textwidth]{example2.png}
\end{center}

$s_1$ : closed, bounded
$s_2$ : closed, unbounded
\[ \text{Arg}(Z) = -\frac{\pi}{4} \]

EXAMPLE $\mathbb{C}$ is both open and closed, unbounded

EXAMPLE

\begin{center}
\includegraphics[width=0.5\textwidth]{example3.png}
\end{center}

neither open nor closed

EXAMPLE $|\text{Arg}(Z)| \leq \frac{3\pi}{2}$ closed, unbounded

\begin{center}
\includegraphics[width=0.5\textwidth]{example4.png}
\end{center}

$\text{Re}Z > 0$
Functions of a Complex variable.

Let $S$ be the domain of

$$ f : S \rightarrow \mathbb{C} $$

The range $R_f$ is the set

$$ R_f = \{ w \in \mathbb{C} : \exists z \in S, f(z) = w \} $$

As a mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$ there exist functions $u$ and $v$ such that

$$ f(z) = u(x, y) + i v(x, y) \quad z = x + iy $$

$$ f(z) = u(r, \theta) + i v(r, \theta) \quad z = re^{i\theta} $$

**EXAMPLE**

$$ f(z) = z^2 $$

$$ f(z) = (x^2 - y^2) + i 2xy $$

$$ f(z) = r^2 \cos 2\theta + i r^2 \sin 2\theta $$

**EXAMPLE**

Square root $f(z) = \sqrt{z}$ where we choose the branch having

$\operatorname{Re} \sqrt{z} > 0$.

In polar with $\theta = \operatorname{Arg} z$

$$ f(z) = \sqrt{z} $$

$$ f(z) = r^{\frac{1}{2}} \cos \left( \frac{\theta}{2} \right) + i r^{\frac{1}{2}} \sin \left( \frac{\theta}{2} \right) $$
**Example**  
\[ f(z) = \frac{z}{z + \overline{z}} \]

The domain of \( f \) excludes the \( x=0 \) axis, and

\[ f(z) = \frac{x + iy}{(x+iy)+(x-iy)} = \frac{1}{2} + \frac{i}{2} \frac{y}{x} \]

\[ u(x,y) \quad v(x,y) \]

**Example**  
\[ f(z) = |z|^2 = (x^2 + y^2) + 0 \cdot i \]

**Example**  
\[ f(z) = \text{Arg} \, z = \theta \]

**Example**  
Rational function

\[ f(z) = \frac{z + 1}{z - 1} \quad z = x + iy \]

then

\[ f(z) = \frac{(z+1)(\overline{z}-1)}{|z-1|^2} = \frac{[(x+1)+iy][(x-1)-iy]}{(x-1)^2 + y^2} \]

Expand and collect real/imaginary parts

\[ f(z) = \frac{x^2 + y^2 - 1}{(x-1)^2 + y^2} + i \frac{-2y}{(x-1)^2 + y^2} \]

\[ u(x,y) \quad v(x,y) \]
Region Mappings

Generally, $f$ maps regions into regions.

EXAMPLE $f(z) = z + 2$

EXAMPLE $f(z) = e^{i \pi z}$

EXAMPLE $f(z) = \bar{z}$
$f(z) = z^2$

$\text{Re } z^2 = x^2 - y^2$

$\text{Im } z^2 = 2xy$
Summary of limits

**Definition 0.1** We say the limit

\[ \lim_{z \to z_0} f(z) = L \]

if every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that

\[ |z - z_0| < \delta \Rightarrow |f(z) - L| < \epsilon \]

\( f(z_0) \) need not be defined at \( z_0 \)

**Definition 0.2** We say \( f(z) \) is continuous at \( z_0 \) if

\[ \lim_{z \to z_0} f(z) = f(z_0) \]

Here, \( f(z_0) \) is defined at \( z_0 \)

All of the main limit properties that apply to real valued functions also apply to complex valued functions. Below is a partial list where it is assumed all the limits exist.

\[
\begin{align*}
\lim_{z \to z_0} af(z) &= a \lim_{z \to z_0} f(z) & a \in \mathbb{C} \\
\lim_{z \to z_0} (f(z) + g(z)) &= \lim_{z \to z_0} f(z) + \lim_{z \to z_0} g(z) \\
\lim_{z \to z_0} f(z)g(z) &= \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z) \\
\lim_{z \to z_0} f(g(z)) &= f(g(z_0))
\end{align*}
\]  

**Theorem 0.3** For \( f : \mathbb{C} \to \mathbb{C} \),

\[ f(z) = u(x, y) + iv(x, y) \]

for some functions \( u, v \). For \( z_0 = (x_0, y_0) \), the function \( f(z) \) is continuous if and only if both \( u(x, y) \) and \( v(x, y) \) are continuous at \( (x_0, y_0) \).
**Example** Prove \( \lim_{z \to z_0} z^2 = z_0^2 \)

Choose \( \varepsilon > 0 \) and define

\[
M = \max \left\{ \frac{|z + z_0|}{|z - z_0|} \mid |z - z_0| < r \right\}
\]

max exists. Then if \( \delta = \frac{\varepsilon}{2M} \) we have

\[
|z^2 - z_0^2| = |z + z_0| |z - z_0| < M |z - z_0| < \frac{\varepsilon}{2}
\]

if \( |z - z_0| < \delta \).

**Example** Show \( f(z) = \frac{\bar{z}}{z} \) not cont. at \( z = 0 \).

\[
f(z) = \frac{x - iy}{x + iy}
\]

\[
\lim_{y \to 0, x=0} f(z) = -1 \quad \text{(A)}
\]

\[
\lim_{x \to 0, y=0} f(z) = +1 \quad \text{(B)}
\]

Since \( \text{(A)} \neq \text{(B)} \), \( \lim_{z \to 0} f(z) \) DNE and \( f \) not cont at \( 0 \).
**Example** \( f(z) = \text{Arg } z \quad \text{discont } x \leq 0, y = 0 \)

![Diagram showing discontinuity on the cut \( \pi \).]

**Example** Principal Branch \( f(z) = \sqrt{z} \)

\[ z = r e^{i \theta}, \quad f(z) = r^{1/2} e^{i \theta/2}, \quad \theta \in (-\pi, \pi] \]

is also discontinuous on \( x \leq 0, y = 0 \)

\[ \lim_{\theta \to \pi} f(z) = i r^{1/2} \]
\[ \lim_{\theta \to -\pi} f(z) = -i r^{1/2} \]

attains values \( \pm i r^{1/2} \) for every \( N_r(z_0). \) Not cont.

**Example** \[ \lim_{z \to 2} \overline{z} = 2 \]

1. \[ |\overline{z} - 2| = |z - 2| = |z-2| < \varepsilon \]

   if \( \delta = \varepsilon \), i.e. if \( |z-2| < \delta = \varepsilon \) then (1) true