Derivative of a Complex function

**Defn** Whenever it exists, the derivative \( f'(z_0) \) of \( f(z) \) at \( z_0 \) is

\[
f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}
\]

**Liebnitz notation** \( w = f(z) \)

\[
\frac{df}{dz} = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

**Defn** A function is **analytic** at \( z_0 \) if there is a neighbourhood \( N_r(z_0) \) such that \( f(z) \) exists at every point \( z \in N_r(z_0) \). Regular and holomorphic are synonyms.

**Defn** A function \( f(z) \) is **entire** if it is analytic at every point.
EXAMPLE \[ f(z) = z^2 \]

\[
\frac{\Delta f}{\Delta z} = \frac{(z+\Delta z)^2 - z^2}{\Delta z} = 2z + \Delta z
\]

Taking the limit \( \Delta z \to 0 \) we obtain

\[ f'(z) = 2z \quad \forall z \in \mathbb{C} \]

Thus \( f(z) = z^2 \) is an \textit{entire} function

EXAMPLE \[ f(z) = \frac{1}{z} \]

\[
\frac{\Delta f}{\Delta z} = \frac{1}{\Delta z} \left( \frac{1}{z+\Delta z} - \frac{1}{z} \right)
\]

\[
= -\frac{1}{z(z+\Delta z)}
\]

Letting \( \Delta z \to 0 \) we find

\[ f'(z) = -\frac{1}{z^2} \quad z \neq 0 \]

The derivative of \( f(z) \) DNE at \( z = 0 \).
Hence \( f(z) \) is also \textit{not} analytic at \( z = 0 \)

nor is it \textit{entire}
**Example** Nonexistence $f(z) = \bar{z}$

\[
\frac{\Delta f}{\Delta z} = \frac{(z + \Delta z) - \bar{z}}{\Delta z} = \frac{\Delta \bar{z}}{\Delta z} = \frac{\Delta x - i \Delta y}{\Delta x + i \Delta y}
\]

From (1) it is evident

\[
\lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z} \quad \text{D.N.E.}
\]

\[
\begin{align*}
\Delta x &= 0 \\
\Delta y &= 0
\end{align*}
\]

\[
\lim_{\Delta x \to 0, \Delta y = 0} \frac{\Delta \bar{z}}{\Delta z} = +1 \quad \text{different so } f(z) \text{ one here.}
\]

\[
\lim_{\Delta y \to 0, \Delta x = 0} \frac{\Delta \bar{z}}{\Delta z} = -1
\]

Conclude: $f(z) = \bar{z}$ is nowhere differentiable
Theorem: If \( f'(z_0) \) exists then \( f(z) \) is continuous at \( z_0 \).

**Proof:**

\[
\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0) = 0
\]

\( f'(z_0) \) exists

from which

\[
\lim_{z \to z_0} f(z) = f(z_0)
\]

This states (and implies)

\[
\begin{align*}
&f(z) \text{ differentiable } \Rightarrow f(z) \text{ continuous} \\
&f(z) \text{ not continuous } \Rightarrow f(z) \text{ not differentiable}
\end{align*}
\]

**Example:**

\( f(z) = \sqrt{z} \)  
Principal Branch

\[
f'(z) = \frac{1}{2\sqrt{z}}
\]

\( f(z) \) DNE along \( \Pi \) since \( f(z) \) not cont on \( \Pi \).

\( f(z) \) DNE at \( z=0 \)

\( f(z) \neq 0 \), the derivative \( f'(z) \) DNE on \( \Pi \) at left.
Definition 0.1 The complex function \( f(z) \) has a derivative \( f'(z) \) at \( z \) if

\[
f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

Definition 0.2 \( f(z) \) is analytic at \( z \) if it is differentiable at every point \( z' \) in some neighbourhood of \( z \). It is entire if it is analytic at all \( z \in \mathbb{C} \).

Theorem 0.3 If \( f(z) \) is differentiable at \( z \) then it is continuous at \( z \).

Remarks: Here we summarize (without proof) many of the derivative properties of complex functions. All look identical to their real counterparts. Below \( c \in \mathbb{C} \) and \( n \) is an integer. Assuming all the derivatives exist

\[
\frac{d}{dz} c = 0 \quad \text{(0.1)}
\]

\[
\frac{d}{dz} z = 1 \quad \text{(0.2)}
\]

\[
\frac{d}{dz} z^n = nz^{n-1} \quad \text{(0.3)}
\]

\[
\frac{d}{dz} [f(z) + g(z)] = f'(z) + g'(z) \quad \text{(0.4)}
\]

\[
\frac{d}{dz} [f(z)g(z)] = f(z)g'(z) + f'(z)g(z) \quad \text{(0.5)}
\]

\[
\frac{d}{dz} \left[ \frac{f(z)}{g(z)} \right] = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2} \quad \text{(0.6)}
\]

\[
\frac{d}{dz} g(f(z)) = g'(f(z))f'(z) \quad \text{(0.7)}
\]
Quotient Rule Proof for $f(z)/g(z)$

Let $h(z) = f(z)/g(z)$. Let $\Delta z \to 0$ in

$$
\frac{\Delta h}{\Delta z} = \frac{h(z+\Delta z) - h(z)}{\Delta z}
$$

$$
= \frac{f(z+\Delta z)}{g(z+\Delta z)} \frac{g(z)}{\Delta z} - \frac{f(z)}{g(z)} \frac{g(z+\Delta z)}{\Delta z}
$$

$$
= \frac{f(z+\Delta z)g(z) - f(z)g(z+\Delta z)}{\Delta z g(z) g(z+\Delta z)}
$$

$$
= \frac{\frac{f(z+\Delta z) - f(z)}{\Delta z} g(z) - f(z) \left( \frac{g(z+\Delta z) - g(z)}{\Delta z} \right) g(z)}{g(z) g(z+\Delta z)}
$$

$$
\to \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}
$$

Chain rule: see text if interested
EXAMPLE Find \( f'(z) \) if \( f(z) = (2iz + 5)^3 \)

Chain rule still applies

\[
\frac{d}{dz} (2iz + 5)^3 = 3(2iz + 5)^2 \frac{d}{dz} (2iz + 5)
\]

\[
f'(z) = 6 i (2iz + 5)^2
\]

EXAMPLE Find \( f'(z) \) if \( f(z) = \frac{z-1}{z+1} \)

Quotient rule still applies

\[
f'(z) = \frac{(z+1) \frac{d}{dz} (z-1) - (z-1) \frac{d}{dz} (z+1)}{(z+1)^2}
\]

\[
f'(z) = \frac{(z+1) - (z-1)}{(z+1)^2}
\]

\[
f'(z) = \frac{2}{(z+1)^2}
\]

Also done via \( f(z) = 1 - 2(z+1)^{-1} \). Then

\[
f'(z) = 2(z+1)^{-2}
\]
Theorem  Cauchy-Riemann equations

Let \( f(z) = u(x,y) + iv(x,y) \) and suppose \( f'(z_0) \) exists at \( z_0 = x_0 + iy_0 \). Then,

\[
(1) \quad f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)
\]

and at \( z_0 \)

\[
(2) \quad u_x = v_y \quad u_y = -v_x
\]

\( \in CR \) equations

Horizontal limit

\[
f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}
\]

as \( \Delta x \to 0 \) this yields (1) in the theorem

\[
(3) \quad f'(z) = u_x(x, y) + iv_x(x, y)
\]

Since \( f'(z) \) exists its value can be found by taking the limit

\( \Delta Z = (\Delta x, 0) \to 0 \)

Path of limit on left.
**Vertical Limit**

\[ f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y)}{i \Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y} \]

Noting \( \frac{1}{i} = -i \) in the first term and the second term is real, then as \( \Delta y \to 0 \)

\[ f'(z) = v_y(x, y) - i u_y(x, y) \]

Since (3) and (4) are equal so must be their real and imaginary parts:

\[ u_x = v_y \quad u_y = -v_x \]

**Remarks**

(i) Can be used to find \( f'(z) \)

(ii) Can be used to prove non-existence of \( f'(z) \)
**Example** \( f(z) = z^2 \quad z = x + iy \)

Here

\[ f(z) = (x^2 - y^2) + 2ixy \]

Since \( f(z) \) exists everywhere

\[ f'(z) = \frac{\partial}{\partial x} (x^2 - y^2) + i \frac{\partial}{\partial y} (2xy) \]

\[ f'(z) = 2x + 2iy \]

\[ f'(z) = 2(x + iy) = 2z \]

Can also verify CR eqns are satisfied

**CR1** \[ u_x = 2x = v_y \] \( \checkmark \)

**CR2** \[ u_y = -2y = -v_x \] \( \checkmark \)
**Example**  How to use CR eqns to show $f'(z_0)$ DNE

$$f(z) = |z|^2 = (x^2 + y^2) + 0 \cdot i$$

If $f'(z)$ exists at $z \neq 0$ then CR-eqns must be satisfied at such $z$. CR eqns are:

$$u_x = \frac{\partial}{\partial x} (x^2 + y^2) = 2x \neq 0 = v_y$$

$$u_y = \frac{\partial}{\partial y} (x^2 + y^2) = 2y \neq 0 = -v_x$$

Since the CR eqns are not satisfied $\forall z \neq 0$ the $f'(z)$ DNE $\forall z \neq 0$.

What about at $z = 0$?

Claim $f'(0) = 0$. By defn of derivative must show (for $z = 0$)

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = 0$$

$$\lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z} = 0$$

which is true since

$$\left| \frac{|\Delta z|^2}{\Delta z} - 0 \right| = |\Delta z| \to 0$$

**Summary**  $f(z) = |z|^2$ is only differentiable at $z = 0$. It is nowhere analytic.
Cauchy Riemann Equivalency

Let \( u, u_x, v_x, u_y \) and \( v_y \) be continuous on some neighbourhood \( N_{r}(Z_0) \) of \( Z_0 \).
Then

\[
f'(Z_0) \text{ exists } \iff \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}
\]

The previous theorem stated if \( f'(Z_0) \) existed, then the CR eqns were necessarily true, which is the \( \Rightarrow \) direction above. To prove the other direction \( \Leftarrow \) one needs to add continuity of \( u, v \) and their derivatives.

**EXAMPLE** Is \( f(z) = i \text{Arg} \, z \) differentiable?

\[
f(z) = 0 + i \arctan \left( \frac{y}{x} \right), \quad \text{Re}(z) > 0
\]

Clearly \( v_y \neq 0, v_x \neq 0 \Rightarrow \) by CR answer is **NO**!

**EXAMPLE** \( f(z) = i \text{Re}(z^2) + \text{Im}(z) \) differentiable?

Since \( z^2 = (x^2 - y^2) + 2ixy \) then

\[
f(z) = y + (x^2 - y^2) \, i = u(x, y) + iv(x, y)
\]

Not differentiable since neither CR eqns true

\[
u_x = 0 \neq v_y \quad \quad u_y = 1 \neq -v_x
\]
EXAMPLE Is there any differentiable function \( f(z) \) with

\[
\lim_{z \to 0} (f(z)) = |z|^2
\]

Any such function would have the form

\[
f(z) = u(x, y) + i \, v(x, y)
\]

\[
f(z) = u(x, y) + i \, (x^2 + y^2)
\]

and would have to satisfy the CR-eqns

\[
\begin{align*}
(1) \quad u_x &= 2y & \quad (2) \quad u_y &= -2x
\end{align*}
\]

Integrating (1) in \( x \)

\[
(3) \quad u = 2xy + \phi(y)
\]

for some unknown fn \( \phi(y) \). Sub (3) into (2)

\[
(4) \quad 2x + \phi'(y) = -2x
\]

Since there is no \( \phi'(y) \) to make (4) true, there is no \( u(x, y) \) that satisfies the CR eqns (1)-(2)

Answer: No
EXAMPLE

Is there any differentiable function $f(z)$ with

$$\Im(f(z)) = x^3 - 3xy^2$$

Any such function would have the form

$$f(z) = u(x, y) + i(x^3 - 3xy^2)$$

and would have to satisfy the CR-eqns

(1) $u_x = -6xy = v_y$

(2) $v_y = 3y^2 - 3x^2 = -v_x$

Integrate (1) in $x$:

(3) $u(x, y) = -3x^2y + \phi(y)$

for some unknown $\phi(y)$. Diff (3) in (2)

$$-3x^2 + \phi'(y) = 3y^2 - 3x^2$$

Hence $\phi(y) = y^3$ and

$$f(z) = (y^3 - 3x^2y) + i(x^3 - 3xy^2)$$

After using

$$x = \frac{1}{2}(z + \bar{z}) \quad y = \frac{1}{2i}(z - \bar{z})$$

much algebra yields $f(z) = i\bar{z}^3$
Cauchy Riemann Eqns in Polar

We seek to re-express CR eqns in polar coordinates. Given

\[ x = r \cos \theta \quad y = r \sin \theta \]

the chain rule yields

\begin{align*}
(1) & \quad u_r = u_x \frac{\partial}{\partial r} (r \cos \theta) + u_y \frac{\partial}{\partial r} (r \sin \theta) \\
(2) & \quad u_\theta = u_x \frac{\partial}{\partial \theta} (r \cos \theta) + u_y \frac{\partial}{\partial \theta} (r \sin \theta)
\end{align*}

Computing the partials in (1)-(2) we can rewrite it as a system:

\begin{equation}
\begin{bmatrix}
    u_r \\
    u_\theta
\end{bmatrix} =
\begin{bmatrix}
    \cos \theta & \sin \theta \\
    -r \sin \theta & r \cos \theta
\end{bmatrix}
\begin{bmatrix}
    u_x \\
    u_y
\end{bmatrix}
\end{equation}

The matrix \( A \) is invertible since \( \det(A) = r \neq 0 \)

Explicitly

\[ A^{-1} = \frac{1}{r} \begin{bmatrix}
    r \cos \theta & -\sin \theta \\
    -r \sin \theta & r \cos \theta
\end{bmatrix} \]

We use this to invert (3) and find \( u_x, u_\theta \) in terms of \( u_r, u_\theta \) etc.
Cartesian CR eqns

\[(CR1) \quad u_x = v_y\]
\[(CR2) \quad u_y = -v_x\]

Using derivative conversion formulae, these become

\[(CR1') \quad \cos \theta u_r - \frac{\sin \theta}{r} u_\theta = \sin \theta v_r + \frac{\cos \theta}{r} v_\theta\]
\[(CR2') \quad \sin \theta u_r + \frac{\cos \theta}{r} u_\theta = -\cos \theta v_r + \frac{\sin \theta}{r} v_\theta\]

Using much algebra, solving for \(u_r\) and \(u_\theta\) we finally get the polar CR eqns

\[
\begin{align*}
ru_r & = v_\theta & \text{CR polar} \\
u_\theta & = -rv_r
\end{align*}
\]

**EXAMPLE**
Is \(f(z) = r \text{Arg} z + iz^2\) differentiable

\[f(z) = \frac{r \theta + i r^2}{u + v}\]

Check CR polar eqns

\[
\begin{align*}
ru_r & = r \theta \neq 0 = v_\theta \quad \text{Not satisfied} \\
u_\theta & = r \neq -2r^2 = -rv_r
\end{align*}
\]

\(f(z)\) is not differentiable anywhere.

**EXAMPLE**
\(f(z) = z^{-1} = \frac{1}{r} \cos \theta + i \left( \frac{-1}{r} \sin \theta \right)\) is differentiable.