Defn \[ e^z = e^x e^{iy} \]

On the real axis \((y = 0)\) this definition agrees with the real exponential function \(e^x\). Thus, \(e^z\) is an extension of \(e^x\) into the complex plane.

As a complex function \(e^z\) obeys many of the same properties as its real counterpart.

1. \[ e^{z_1} e^{z_2} = e^{z_1 + z_2} \]
2. \[ e^{-z} = \frac{1}{e^z} \]
3. \[ e^{z + 2\pi i} = e^z \text{ periodic} \]
4. \[ \frac{d}{dz} e^z = e^z \text{ (entire)} \]

Property (3) is unusual as it says \(e^z\) is a periodic function but with complex period.

![Diagram](image-url)
Proof of (1)

\[ e^{z_1} e^{z_2} = (e^{x_1+i y_1})(e^{x_2+i y_2}) \]

\[ = (e^{x_1} e^{x_2})(e^{i y_1} e^{i y_2}) \]

\[ = e^{x_1+x_2} e^{i (y_1+y_2)} \]

\[ = e^{z_1+z_2} \]

Proof of (4)

Given the defn of \( e^z \)

\[ f(z) = e^z = \frac{e^x \cos y + i e^x \sin y}{u + v} \]

It is easy to verify \( e^z \) is entire via CR-eqns

\[ u_x = v_y \quad u_y = -v_x \quad \checkmark \]

Hence

\[ f'(z) = u_x + i v_x \]

\[ f'(z) = u + i v \]

\[ f'(z) = f(z) \quad \square \]