$$\boxed{\underline{\text{Defn}} \qquad e^z \equiv e^x e^{iy}}$$

On the real axis ($y=0$) this definition agrees with the real exponential function e^x. Thus, e^z is an extension of e^x into the complex plane.

As a complex function e^z obeys many of the same properties as its real counterpart.

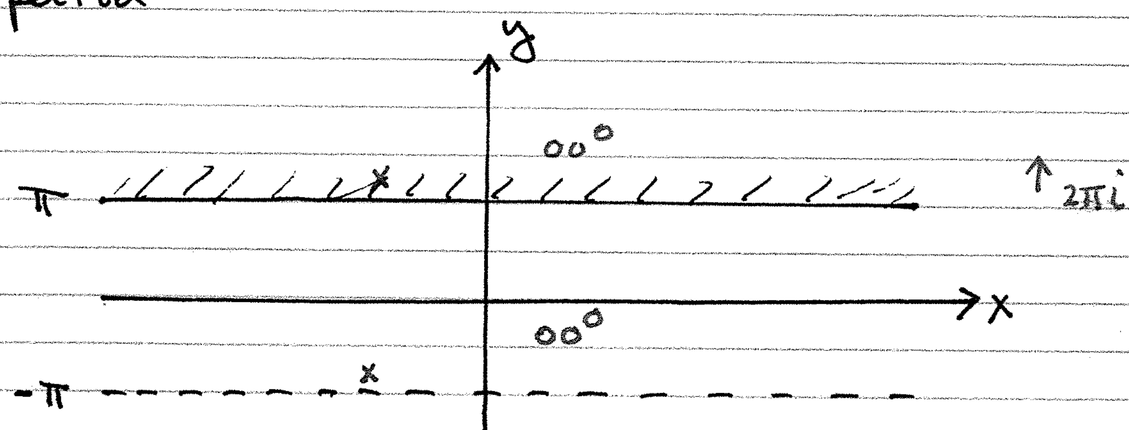
(1) $\quad e^{z_1} e^{z_2} = e^{z_1+z_2}$

(2) $\quad e^{-z} = \dfrac{1}{e^z}$

(3) $\quad e^{z+2\pi i} = e^z \qquad$ periodic

(4) $\quad \dfrac{d}{dz} e^z = e^z \qquad$ (entire)

Property (3) is unusual as it says e^z is a periodic function but with complex period



Proof of (1)

$$e^{z_1} e^{z_2} = (e^{x_1} e^{iy_1})(e^{x_2} e^{iy_2})$$
$$= (e^{x_1} e^{x_2})(e^{iy_1} e^{iy_2})$$
$$= e^{x_1+x_2} e^{i(y_1+y_2)} \quad \text{proved earlier}$$
$$= e^{z_1+z_2}$$

Proof of (4)

Given the defn of e^z

$$f(z) = e^z = \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$$

It is easy to verify e^z is entire via CR-eqns

$$u_x = v_y \qquad u_y = -v_x \qquad \checkmark$$

Hence

$$f'(z) = u_x + i v_x$$

$$f'(z) = u + i v$$

$$f'(z) = f(z) \qquad \square$$