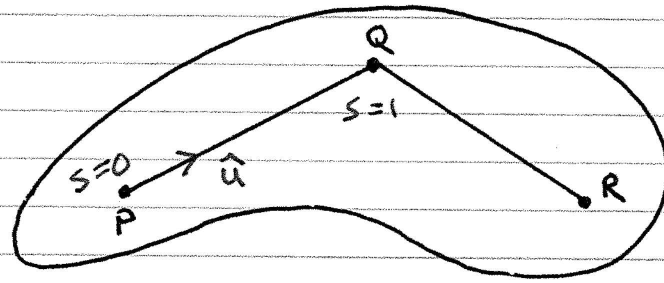
Theorem Let D be open connected.

$$f'(z) = 0 \quad \forall z \in D \quad \Rightarrow \quad f(z) = c \quad \text{constant}$$

Pf/ CR-eqns $\Rightarrow$ $u_x = u_y = v_x = v_y = 0 \quad \forall z \in D$.

Now show $f(z)$ constant on line segments



$$\hat{u} = \frac{\vec{PQ}}{|\vec{PQ}|}$$

$$|\hat{u}| = 1$$

Parametrize segment PQ in $s \in (0,1)$

$$U(s) \equiv u(x(s), y(s)) \qquad \text{real part on } PQ$$

$$\frac{dU}{ds} = \nabla u(x,y) \cdot \hat{u} \qquad \text{directional deriv.}$$

$$\frac{dU}{ds} = 0 \qquad \text{by CR eqns}$$

Hence u constent on all straight lines hence line segments.

Argument the same for imag part $V(s) = v(x(s), y(s))$.

$$u = a \qquad v = b \qquad \text{on } D$$

Hence

$$f(z) = c = a + ib \qquad \forall z \in D \qquad \Box$$

Harmonic Functions

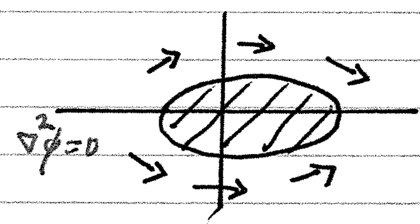
Defn: $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is harmonic on $S \subset \mathbb{R}^2$ if

$$(1) \qquad \nabla^2 \phi \equiv \phi_{xx} + \phi_{yy} = 0 \qquad \forall \vec{x} \in S$$

Remarks: The symbol $\nabla^2\phi$ is called the Laplacian operator as in

$$\nabla^2(x^2+y^3) = 2 + 6y$$

Physics is replete with harmonic functions. Two quick examples

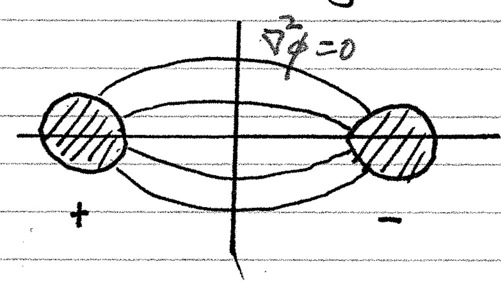


Fluid velocity around a solid

$\vec{v} = \nabla\phi$

where $\phi(x,y)$ is harmonic.

In electromagnetism:



Charged bodies, with no other sources between

Electric fied $\vec{E} = -\vec{\nabla}\phi$ where ϕ is harmonic

Theorem If $f(z) = u(x,y) + i v(x,y)$ is analytic on S then both u and v are harmonic functions on S.

Pf/ Just show $\nabla^2 u = 0$. $v(x,y)$ proof similar. Since $f(z)$ analytic, u and v satisfy the CR-eqns

$$(1) \quad u_x = v_y$$

$$(2) \quad u_y = -v_x$$

Differentiate (1) in x, then use (2).

$$u_{xx} = v_{yx} = v_{xy} \overset{(2)}{=} -u_{yy} \Rightarrow u_{xx} + u_{yy} = 0 \quad \square$$

Remark: We have tacitly assumed all second partials exist and are continuous. This is proven later where if $f(z)$ is analytic, not only does $f'(z)$ exist on S but so does $f^{(n)}(z)$ for all $n = 1, 2, \ldots$

Defn: Two harmonic functions u, v are <u>conjugate</u> if and only if they satisfy the CR-eqns.

EXAMPLE $u(x,y) = x^2 - y^2$ $v(x,y) = e^x \cos y$

It is easy to verify $\nabla^2 u = 0$ and $\nabla^2 v = 0$ but u, v do not satisfy CR-eqns. For one

$$u_x = 2x \neq -e^x \sin y$$

EXAMPLE Find $v(x,y)$ conjugate to

$$u(x,y) = x(2y+1)$$

Using the CR eqns

$$(1) \quad v_y = u_x = 2y + 1$$

$$(2) \quad v_x = -u_y = -2x$$

Integrate (1) in y to get

$$(3) \quad v(x,y) = y^2 + y + \phi(x)$$

Then use (3) in (2)

$$v_x = \phi'(x) = -2x \quad \Rightarrow \phi = -x^2$$

Conclude

$$v(x,y) = (y^2 - x^2) + y$$

Can show these are real/imag. parts of

$$f(z) = -iz^2 + z$$