1. [5pt] Define the (ordered) set $T := \{1, x, x^2\}$ in $L^2[0, 1]$ with the weighted inner product

$$<f, g> = \int_0^1 x^k f(x)g(x) \, dx.$$  

Use the Gram Schmidt procedure to find an orthogonal basis for $\text{span}(T)$.

2. [5pt] Let $S = \mathbb{R}^3$ and $f = (4, 1, -1)$. Find the coordinate $\vec{\alpha}$ of $f$ relative to $T = \{\phi_1, \phi_2, \phi_3\}$ where $\phi_1 = (1, 1, 0)^T$, $\phi_2 = (0, 1, 1)^T$, $\phi_3 = (1, 0, 1)^T$.

3. [5pt] Orthogonally diagonalize

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

4. [5pt] Let $P(x)$ be the projection of $x \in \mathbb{R}^3$ onto $T = \text{span}\{\phi_1, \phi_2\}$ where $\phi_1 = (1, 0, -1)^T$, $\phi_2 = (1, 1, 1)^T$. Find the matrix $A$ such that $P(x) = Ax$. Then find all the eigenvalues of $A$. Is $A$ invertible? Note $\phi_k$ aren’t normalized.

5. [7pt] Consider the eigenvalue problem

$$A(\varepsilon)x(\varepsilon) = \lambda(\varepsilon)x(\varepsilon)$$

where $0 < \varepsilon \ll 1$ is a small parameter and

$$A(\varepsilon) = \begin{pmatrix} 1 - \varepsilon & 2 \\ 8 & 1 + 2\varepsilon \end{pmatrix}.$$  

Using the expansions

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \cdots$$

$$x(\varepsilon) = x_0 + \varepsilon x_1 + \cdots$$

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon \lambda_1 + \cdots$$
and the Fredholm alternative, find $\lambda_1$ and $x_1$ for both of the eigenvalues of $A$. Note $x_1$ is not unique.

6. [8pt] Consider the perturbed algebraic problem

\begin{align*}
x - 2y &= \epsilon(x^2 - y) \\
2x - 4y &= \epsilon(x - y^2)
\end{align*}

Use the Fredholm Alternative to find (all) $x_0, y_0$ in the following expansions of the roots of the system above:

\begin{align*}
x(\epsilon) &= x_0 + \epsilon x_1 + \cdots \\
y(\epsilon) &= y_0 + \epsilon y_1 + \cdots
\end{align*}

7. [5pt] Determine the normal equations for and least squares solution of

$$Ax = b$$

where

$$A = \begin{bmatrix}
1 & 0 \\
0 & 2 \\
0 & 1 \\
0 & 1
\end{bmatrix}, \quad b = \begin{pmatrix}
1 \\
1 \\
1 \\
2
\end{pmatrix}$$