Math 560: Assignment 2: Due Oct 13, 2016

1. [10] Define
$$T = \{\phi_n\} \equiv \{\sin nx\}_{n=1}^{\infty}$$
T is a complete orthogonal set for $L^2[0,\pi]$. Next, define the kernel
$$k(x,y) = \sum_{n=1}^{\infty} \frac{\phi_{n+1}(x)\phi_n(y)}{n^2} \tag{1}$$
and corresponding integral operator
$$Ku = \int_0^{\pi} k(x,y)u(y)dy \tag{2}$$

   i) Prove K is a bounded operator. For this you need to show
   $$M \equiv \int_0^{\pi}\int_0^{\pi} k^2(x,y)\ dy\ dx < \infty$$
   noting $\| Ku \|^2 \leq M \| u \|^2$.

   ii) Is K self adjoint?

   iii) Show that $N(K) = \{0\}$.

   Note: It is also possible to show K has no eigenvalues!!

2. [20] Each of the following operators L are defined on $L^2[a,b]$ under the usual L^2 inner product. For each operator and associated boundary conditions determine the adjoint operator L^* and its domain. Decide which (if any) operators are self adjoint.

   (i) $Lu = x^2u'' - xu',\ x \in (1,2)$ $\quad u(1) = 0,\ u(2) - 2u'(2) = 0$

   (ii) $Lu = x(u''' - u'),\ x \in (0,1)$ $\quad u(0) = u'(0) = 0,\ u''(1) - u'(1) = 0$

   (iii) $Lu = u''(x) + \int_0^1 xyu(y)dy$ $\quad u(0) = u'(0) = 0$

   (iv) $Lu = \int_0^{x^2} u(y)\ dy$ $\quad D(L) = L^2(0,\infty)$

3. [5] Let $Lu = u''(x), x \in (0,1)$ and

$$D(L) = \{u : \| u \| < \infty, \ \| Lu \| < \infty, u(0) = u'(1) = 0\} \ . \quad (3)$$

For each of the following L^2 and *Sobolev* inner products determine the adjoint operator L^* and its domain $D(L^*)$.

$$< u, v > \ = \ \int_0^1 u(x)v(x)dx \quad (4)$$

$$< u, v > \ = \ \int_0^1 \left(u(x)v(x) + u'(x)v'(x)\right) dx \quad (5)$$

4. [5] Recall the $\ell^2(\mathbb{R})$ Hilbert space

$$\ell^2(\mathbb{R}) = \{\{x_n\} : \sum_{n=1}^{\infty} x_n^2 < \infty, x_n \in \mathbb{R}\}$$

with inner product

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n$$

Define the operator

$$Lx \equiv (x_2, x_1, x_3, x_4, x_5, \ldots)$$

a) Prove L is bounded
b) Well define the set of “eigenvectors“ with eigenvalue $\lambda = 1$.
c) Is L self adjoint?