Math 560: Assignment 3: Due Nov 3-7, 2016

1. [10] The following operator is self adjoint on $D(K) = L^2[0,1]$:

$$Ku \equiv \int_0^1 x^2 y^2 u(y)\, dy$$

   a) Find the nonzero eigenvalue λ_1 of K and carefully define $N(K - \lambda_1 I)$

   b) Use the Fredholm alternative to find all values of α such that

$$(K - \lambda_1 I)u = 5x^2 + 2x + \alpha$$

   has a solution.

   c) Consider the perturbed nonlinear integral equation:

$$(K - \lambda_1 I)u = \epsilon\ (u^3 - 9) \qquad 0 \leq \epsilon \ll 1$$

   with expansion $u(x) = u_0(x) + \epsilon u_1(x) + O(\epsilon^2)$. Use the Fredholm alternative to show the $O(\epsilon)$ problem for u_1 has a solution only for a unique $u_0 \in N(K - \lambda_1 I)$.

2. [5] An operator L is defined by

$$Lu \equiv u'' \qquad , \qquad D(L) = \{u \in C^2[0,7] : u'(0) = u'(7) = 0\}$$

   Use Fredholm's alternative to prove $Lu = f$ has a solution only if the average of $f(x)$ is zero on $[0,7]$.

3. [5] Let K be a self adjoint Hilbert-Schmidt operator on $L^2[a,b]$ with eigenvalues λ_n and associated (complete) set of eigenfunctions $\{\phi_n\}$. Next define the N^{th}degree polynomial

$$P_N(z) = 1 + z + \cdots \frac{1}{N!} z^N$$

   and the operator $P_N(K)$ defined by

$$P_N(K)u = (I + K + \cdots \frac{1}{N!} K^N)u$$

   For each u we have

$$P_N(K)u = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

Find b_n for finite N and (providing the limits exist) as $N \to \infty$. In this manner the operator e^K can be defined.

4. [8] Use the properties of distributions to show the following

$$\begin{aligned} \sin x\ \delta'(x) &= -\delta(x) \qquad (1) \\ \delta\left(\frac{x+1}{2}\right) &= 2\delta(x+1) \qquad (2) \end{aligned}$$

5. [7] Here consider a restricted set of test functions:

$$D_0 \equiv \{\phi \in D : \phi(0) = 0\}$$

recalling all such $\phi(x)$ are smooth (convergent Taylor series). Carefully show the derivative

$$\left\langle Pv'\left(\frac{1}{x}\right), \phi\right\rangle = \left\langle Pv\left(\frac{1}{x}\right), -\phi'\right\rangle = \left\langle -Pv\left(\frac{1}{x^2}\right), \phi\right\rangle$$

In particular, explain why the Principal integral on the right side is defined for all $\phi \in D_0$.