Math 560-561: Homework 5.

Due: Thursday, Feb 2, 2017.

1. [15] For the following problems, there is an extrema $y \in \mathcal{A}$ such that the first variation $\delta J(y,h) = 0$ for all admissible variations $h \in \mathcal{A}^*$. In each case $y(x)$ solves a boundary value problem. Some boundary conditions help define $\mathcal{A}$ and others are derived (or natural) boundary conditions. The latter may be complicated expressions. For each functional, derive and clearly state the boundary value problem. Do not attempt to solve the boundary value problems.

   a)

$$J(y) = \int_1^2 \left(2y^2 + x^2 y'^2\right) dx \qquad (1)$$

$$\mathcal{A} = \{y \in C^2[1,2] : y(1) = 1\} \qquad (2)$$

   b)

$$J(y) = y(1)^2 + \int_1^2 \sqrt{1+y'^2}\,dx \qquad (3)$$

$$\mathcal{A} = \{y \in C^2[1,2] : y(2) = 5\} \qquad (4)$$

   c)

$$J(y) = \int_1^2 xy'y''\ dx \qquad (5)$$

$$\mathcal{A} = \{y \in C^4[1,2] : y(1) = 0\} \qquad (6)$$

2. [10] Consider the functional

$$J(y) = \int_a^b L(y, y')\ dx$$

   where

$$L(y, y') = y^2(1 - y')^2$$

   a) Use a first integral to find the general solution of the Euler Lagrange equations associated with extremizing $J(y)$. The substitution $u = y^2$ may be helpful though not necessary.

b) Define the admissible sets

$$\begin{aligned}\mathcal{A}_1 &= \{y \in C^2[-1,1] \;:\; y(-1)=0, y(1)=1\} \\ \mathcal{A}_2 &= \{y \in C^2[1,2] \;:\; y(1)=1, y(2)=3/\sqrt{2}\}\end{aligned}$$

Over which of these admissible sets does the functional $J(y)$ have an extrema and why? I will tell you that one of the extrema is defined on $\mathcal{A}_k$ while the other is not.

3. [10] Consider the functional

$$J(y) = \int_0^L L(y, y')\; ds$$

where

$$L(y,y') = y(s)\sqrt{1 - y'(s)^2}$$

and s is the arclength parameter for some curve $(x(t), y(t))$.

a) Show that the function which maximizes $J(y)$ must satisfy

$$\frac{y(s)}{\sqrt{1-y'(s)^2}} = r$$

for some constant r. Then use $y(s) = r\sin\theta(s)$ to show $\theta = s/r + c$ where c is some other constant.

b) Show the maximizer in $\mathcal{A} = \{y(s) \in C^1[0,L] \;:\; y(0)=0, y(L)=0\}$ is $y(s) = r\sin(s/r)$ where $r = L/\pi$.

c) Show the maximizer is a semicircle of radius r using

$$x(s) = \int_0^s \sqrt{1-y'(t)^2}\; dt$$

The basic idea here is to show $(x(s), y(s))$ is a parametrization of a circle centered at $(r, 0)$.

4. [5] Geodesics on a paraboloid. Curves Γ on the paraboloid $z = x^2 + y^2$ generally have a parametrization $\Gamma = (X(t), Y(t), Z(t))$. We shall assume that, on geodesics, $r = R(\theta)$ where (r, θ, z) are cylindrical coordinates and R is some function. Geodesics must then minimize a functional

$$J(R) = \int_{\theta_1}^{\theta_2} L(R, R')\; d\theta$$

for some Lagrangian L.

a) Show $L(R, R') = \sqrt{R^2 + (1+4R^2)R'^2}$.

b) Find a first order differential equation which the minimizer $R(\theta)$ must minimize. Do not attempt to solve the differential equation.