Math 560-561: Homework 6.
Due: Thursday, February 23, 2017.

1. [10] Define the functional

$$J(y) = \int_0^1 L(y, y')\; dx = \int_0^1 y^2 + y'^2\; dx$$

on $\mathcal{A}$ where

$$\mathcal{A} = \left\{ y \in C^2[0,1] \;:\; y(0) = 0, y(1) = 1, \int_0^1 y^2\; dx = 5 \right\}$$

The modified Lagrangian for the problem is

$$M(y, y', \lambda) \equiv L(y, y') - \lambda K(y, y')$$

where the constraint Lagrangian is $K = y^2$.

a) Find families $y_\lambda(x)$ of solutions to the modified Euler Lagrange equations which satisfy all but the integral constraint.

b) Find a function $F(\lambda)$ such that $y_\mu(x)$ is the extrema of J if $F(\mu) = 5$. Numerically estimate all such values of μ.

2. [10] Define the functional

$$J(y) = \int_0^t L(y, y')\; dx = \int_0^t \sqrt{y^2 + y'^2}\; dx$$

on $\mathcal{A}$ where

$$\mathcal{A} = \left\{ y \in C^2[0,t] \;:\; y(0) = 1\;,\; f(t, y(t)) = 0 \right\}$$

Here $x = t$ is a variable endpoint and $f(t, y) = y + t - 2$.

a) Determine the variable endpoint boundary value problem the extrema of J must satisfy. In particular, calculate the transversality condition for the problem.

b) Show that general solution of the Euler Lagrange equations is

$$y(x) = \alpha \sec(x + \phi)$$

where α, ϕ are constants. Define the system $F_i(\alpha, \phi, t) = 0, i = 1, 2, 3$ whose solutions (α, ϕ, t) complete the extremal solution.

3. [10] Prove the extrema of J below is a unique minimizer and compute it.

$$\begin{aligned} J(y) &= \int_1^2 \left(2y^2 + x^2 y'^2\right)\, dx \\ \mathcal{A} &= \{y \in C^2[1,2], y(1) = 1, y(2) = \frac{1}{4}\} \end{aligned}$$

4. [10] Let Ω be the annular region $\Omega = \{(r, \theta) : 1 < r < 2\ ,\ 0 \leq \theta < 2\pi\}$ and define the following functional with associated admissible set

$$\begin{aligned} J(u) &= \int_\Omega (u_x^2 + u_y^2) dA \\ \mathcal{A} &= \{u \in C^2(\bar{\Omega}) : u|_{\partial\Omega} = f(x)\} \end{aligned}$$

a) Carefully derive the Euler Lagrange equations for $u(x, y)$.

b) Prove extrema are minima

c) Find the minimizer for (polar)

$$f(r, \theta) = \begin{cases} 0 & on \quad r = 1 \\ \alpha & on \quad r = 2 \end{cases}$$

5) [10] A pendulum (moving only in the xy-plane) has its pivot attached to a mass m_1 which slides without friction horizontally along the x axis. It's position is $(x_1, 0)$. The pendulum of mass m_2 has a fixed length r and makes angle θ with respect to the y axis downward. It's position is $(x_2, y_2) = (x_1 + r \sin\theta, r\cos\theta)$. The total kinetic energy of the system is:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

Since only gravity acts on the system, $U = -m_2 g y_2$

a) For the coordinates $q = (x_1, \theta)$, derive the Lagrangian $L = L(q, \dot{q})$.

b) Write out the Euler Lagrange equations as two coupled second order equations for q.

c) Let $\theta = \epsilon\phi$ in the EL equations of part b). Derive the leading small $(0 < \epsilon \ll 1)$ amplitude equations for the system (it should be linear). Find the general solution for $x_1(t)$ and $\phi(t)$ in terms of constants of integration to be found from initial conditions. State the frequency ω of $\phi(t)$. Is the frequency higher/lower than that $\omega_0 = \sqrt{g/r}$ when m_2 is fixed? Lastly, is the motion of the system periodic?