1 An Introduction to Asymptotics

Many functions $u(\epsilon)$ do not have regular expansions in ϵ. However, they may still have other kinds of expansions in ϵ. Some such series *representations* may converge or even diverge as $\epsilon \to 0$. A few simple examples include:

$$\begin{aligned} u_1(\epsilon) &= 1 + \sqrt{\epsilon} + \epsilon + \epsilon^{3/2} + \cdots \\ u_2(\epsilon) &= 1 + \epsilon \ln \epsilon + \epsilon^2 (\ln \epsilon)^2 + \cdots \\ u_3(\epsilon) &= \epsilon^{-1} + 1 + \epsilon + \epsilon^2 \cdots \end{aligned}$$

Here the series defining u_1 is convergent but $u_1'(0)$ is undefined. For small ϵ the series defining u_2 is clearly convergent and is readily seen that

$$u_2(\epsilon) = \frac{1}{1 - \epsilon \ln \epsilon}$$

Nevertheless, it has no regular expansion about $\epsilon = 0$. Lastly, for each fixed $\epsilon \in (0,1)$ the series representation of u_3 converges but it is not a regular expansion. Indeed, it has the singular behavior $u_3 \to \infty$ as $\epsilon \to 0$.

Though much of what follows can be generalized to expressions valid as $\epsilon \to \epsilon_0$ we shall, w.l.o.g., restrict ourselves to the $\epsilon \to 0$ case. First we define notations for neighborhoods and punctured neighborhoods:

$$\begin{aligned} N_r(0) &\equiv \{\epsilon : 0 \leq |\epsilon| < r\} \\ \bar{N}_r(0) &\equiv \{\epsilon : 0 < |\epsilon| < r\} \end{aligned}$$

For the cases $\epsilon > 0$ we shall use I to denote the open interval $I = (0, r)$. Given these conventions we have:

Definition 1. $u = O(\phi)$ *if* $\exists\ M, r$ *independent of* ϵ *such that*

$$|u(\epsilon)| \ \leq M \ |\phi(\epsilon)| \qquad , \qquad \forall \epsilon \in \bar{N}_r(0)$$

Definition 2. $u = o(\phi)$ *if* $\forall \delta > 0\ \exists\ r$ *independent of* ϵ *such that*

$$|u(\epsilon)| \ \leq \delta \ |\phi(\epsilon)| \qquad , \qquad \forall \epsilon \in \bar{N}_r(0)$$

Definition 3. *A notational definition:*

$$u = o(\phi) \qquad \Leftrightarrow \qquad u \ll \phi$$

For functions that are well behaved on $\bar{N}_r(0)$ as $\epsilon \to 0$ the big "O" definition is best understood as the ratio $|u/\phi|$ is bounded. Note however this need not mean that either u, ϕ are bounded or approach some limiting value as $\epsilon \to 0$. Consider for instance the statements

$$\begin{aligned} \sin\left(\frac{1}{\epsilon}\right) &= O(1) \\ \frac{1}{\epsilon} &= O\left(\frac{1}{\epsilon^2}\right) \end{aligned}$$

For both of these $M = 2$ is sufficient. Also, $u = O(\phi)$ does not mean that u and ϕ are of the "same magnitude" but more like the "order" of u is at most that of ϕ. A simple example to illustrate this is:

$$\epsilon^n = O(1) \quad , \quad \forall n \geq 0$$

In contrast, the little "o" statement $u \ll \phi$ implies the ratio $|u/\phi|$ can be made arbitrarily small as $\epsilon \to 0$. A very useful Theorem regarding such ratios is:

Theorem 1. *Providing the limits exist*

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{u(\epsilon)}{\phi(\epsilon)} &\neq 0 \qquad \Rightarrow \quad u = O(\phi) \quad \textit{and} \quad \phi = O(u) \\ \lim_{\epsilon \to 0} \frac{u(\epsilon)}{\phi(\epsilon)} &= 0 \qquad \Rightarrow \quad u = o(\phi) \quad \textit{or} \quad u \ll \phi \end{aligned}$$

Generally speaking Big "O" is a statement regarding upper bounds of the relative sizes of u and ϕ whereas little "o" is a statement about ordering growth/decay rates of u and ϕ as $\epsilon \to 0$. In many texts the limits above are used to define O and o.

Definition 4. *A function $u(\epsilon)$ is said to be transcendentally small in ϵ if for all integers $n \geq 0$*

$$u(\epsilon) = o(\epsilon^n)$$

in which case $u(\epsilon)$ is a T.S.T. for Transcendentally Small Term.

The classic example of a transcendentally small function is

$$u(\epsilon) = \exp\left(-\frac{1}{\epsilon}\right)$$

which can be proven by showing

$$\lim_{\epsilon \to 0^+} f(\epsilon) = \lim_{\epsilon \to 0^+} \epsilon^{-n} \exp\left(-\frac{1}{\epsilon}\right) = 0$$

This limit follows since $\ln f(\epsilon) \to -\infty$.

Now suppose $u_1(\epsilon)$ and $u_2(\epsilon)$ are given by:

$$\begin{aligned} u_1(\epsilon) &= \frac{1}{1-\epsilon} \\ u_2(\epsilon) &= \frac{1}{1-\epsilon} + \exp\left(-\frac{1}{\epsilon}\right) \end{aligned}$$

Despite the fact that $u_1(\epsilon)$ has a regular expansion in ϵ the difference $u_2 - u_1$ does not. As far as regular expansions go, these two functions cannot be distinguished.

Example 1. *Here are a few examples of o and O useage:*

$$\begin{aligned} \epsilon^n &\ll \epsilon^m \quad , \quad m < n \\ \cot \epsilon &= O\left(\epsilon^{-1}\right) \\ \sin \epsilon &= O(\epsilon) \\ \epsilon^\alpha \ln \epsilon &\ll 1 \quad , \quad \alpha > 0 \\ \exp\left(-\frac{1}{\epsilon}\right) &\ll \epsilon \ll \epsilon \ln \epsilon \\ f(\epsilon) = \int_0^\infty \frac{e^{-t}}{1+\epsilon t} dt &= O(1) \end{aligned}$$

In particular, the latter follows since

$$f(\epsilon) < \int_0^\infty e^{-t} dt = O(1)$$

Big O and o also have a number of relations with limited algebraic properties. We summarize a few below without proof. w.l.o.g all are true as $\epsilon \to 0$.

Theorem 2. *Let $f_1 = O(\phi_1)$ and $f_2 = O(\phi_2)$. Then*

$$\begin{aligned} f_1 + f_2 &= O(|f_1| + |f_2|) \\ f_1 f_2 &= O(f_1 f_2) \end{aligned}$$

For the results above with $f = f_1, g = f_2$ we adopt the notation

$$\begin{aligned} O(f) + O(g) &= O(|f| + |g|) \\ O(f)O(g) &= O(fg) \end{aligned}$$

Adopting such notation

$$\begin{aligned} O(f)o(g) &= o(f)o(g) = o(fg) \\ O(O(f)) &= O(f) \\ o(O(f)) &= O(o(f)) = o(o(f)) = o(f) \end{aligned}$$

Theorem 3. *O is <u>reflexive</u> and <u>transitive</u> but not symmetric:*

$$\begin{aligned} f &= O(f) \\ f = O(g) \ \ \text{and} \ \ g = O(h) \ \ \Rightarrow \ \ f &= O(h) \end{aligned}$$

A counter-example for non-symmetry is $\epsilon = O(1)$ but $1 \neq O(\epsilon)$.

Theorem 4. *o is <u>transitive</u> but not reflexive or symmetric:*

$$f = o(g) \ \ \text{and} \ \ g = o(h) \ \ \Rightarrow \ \ f = o(h)$$

Reflexivity is clearly not true since $1 \neq o(1)$. Morever, even though $\epsilon = o(1)$, $1 \neq o(\epsilon)$.

1.1 Asymptotic Expansions

We wish to generalize the notion of an approximating expansion. Toward this end we first make the following definition:

Definition 5. *$\phi(\epsilon)$ is an asymptotic approximation to $u(\epsilon)$ if*

$$u = \phi + o(\phi)$$

Equivalently we may write

$$u \sim \phi \qquad\qquad \epsilon \to 0$$

or that u is asymptotic to ϕ.

If two functions u and ϕ are asymptotic to each other their difference need not approach zero as $\epsilon \to 0$! The definition is equivalent to

$$\frac{u - \phi}{\phi} \ll 1 \tag{1}$$

which in no way says ϕ is small. A very simple example of this would be

$$u = \frac{1}{\epsilon} + \frac{1}{\sqrt{\epsilon}} \qquad\qquad \phi = \frac{1}{\epsilon}$$

Here $u \sim \phi$ but clearly $u - \phi \to \infty$ as $\epsilon \to 0$. Given (1), the statement $u \sim \phi$ implies u and ϕ are <u>*relatively*</u> close but not necessarily absolutely close.

Theorem 5. *$\sim$ is an equivalence relation*

$$\begin{aligned} u &\sim u \\ u \sim v &\Rightarrow v \sim u \\ u \sim v \text{ and } v \sim w &\Rightarrow u \sim w \end{aligned}$$

Proof 1. *The latter is true since*

$$\frac{u - w}{w} = \underbrace{\left(\frac{u - v}{v}\right)}_{\to 0} \underbrace{\left(\frac{v}{w}\right)}_{\to 1} + \underbrace{\left(\frac{v - w}{w}\right)}_{\to 0}$$

Definition 6. $\{\phi_n(\epsilon)\}$ *is said to be an asymptotic sequence if*

$$\phi_{n+1} \ll \phi_n \qquad \qquad \forall n$$

Since $\ll$ is transitive, elements of an asymptotic sequence are asymptotically ordered. For example $\phi_2 \ll \phi_{10} \ll \phi_{20}$.

Definition 7. *If* $\{\phi_n(\epsilon)\}$ *is an asymptotic sequence and*

$$u(\epsilon) = \sum_{k=1}^{n} a_k \phi_k(\epsilon) + o(\phi_n) \tag{2}$$

for some constants a_k *then* $S_n(\epsilon) = \sum_{k=1}^{n} a_k \phi_k(\epsilon)$ *is an* n*-term asymptotic approximation of* u

The functions $\phi_k(\epsilon)$ are sometimes called gauge, scale or basis functions. Also, sometimes one merely writes:

$$u(\epsilon) \sim \sum_{k=1}^{n} a_k \phi_k(\epsilon)$$

Such expansions are in no way unique.

Example 2. *Asymptotic expansion examples*

$$u(\epsilon) = 1 + \sqrt{\epsilon} + 2\epsilon^{3/2} + 3\epsilon^{5/2} + O(\epsilon^3)$$

$$u(\epsilon) = 1 + \exp(-1/\epsilon) + \exp(-2/\epsilon) + \exp(-3/\epsilon) + o\left(\exp(-3/\epsilon)\right)$$

$$u(\epsilon) = \frac{1}{\sqrt{\epsilon}} + \frac{1}{\epsilon \ln \epsilon} + \cos\epsilon + O(\sin\epsilon)$$

Example 3. *Nonuniqueness of representation*

$$\ln(1+\epsilon) \sim \epsilon + O(\epsilon^2)$$

$$\ln(1+\epsilon) \sim \sin\epsilon + O(\epsilon^2)$$

Example 4. *Convergent asymptotic series: a most common example of a convergent asymptotic series is the Taylor series:*

$$u(\epsilon) = u(0) + u'(0)\epsilon + \frac{1}{2!}u''(0)\epsilon^2 + O(\epsilon^3)$$

Example 5. *Divergent asymptotic series: It is not necessarily true that the sum S_n in*

$$u \sim S_n(\epsilon) = \sum_{k=1}^{n} a_k \phi_k(\epsilon)$$

converges as $n \to \infty$. To illustrate this, consider

$$I(\epsilon) \equiv \int_0^\infty \frac{e^{-t}}{1+\epsilon t}\, dt$$

If we expand the integrand we have the exact expression

$$I = \sum_{k=0}^{n-1} (-1)^k \epsilon^k \int_0^\infty e^{-t} t^k \, dt + R_n$$

or,

$$I = \sum_{k=0}^{n-1} (-1)^k \epsilon^k k! + R_n$$

is a divergent series (in n) even though the remainder term

$$R_n = \int_0^\infty \frac{e^{-t}(-\epsilon t)^n}{1+\epsilon t}\, dt \quad = \quad O(\epsilon^n)$$

Despite the fact that some asymptotic series diverges, their truncated series are often close in an absolute sense. Case in point, even with a modestly large $\epsilon = 0.1$ one can verify that $|I(\epsilon) - S_{20}(\epsilon)| < 0.008$.

Regardless of whether the asymptotic series converges or not the coefficients in the expansion can always be "recovered". For example, if one merely knew a_k in

$$u(\epsilon) = \sum_{k=0}^{n} a_k \epsilon^k + O(\epsilon^{n+1})$$

then through repeated differentiations

$$a_n = \frac{1}{n!} u^{(n)}(0)$$

which is just the Taylor series coefficients. More generally, suppose

$$u(\epsilon) \sim \sum_{k=1}^{n} a_k \phi_k(\epsilon)$$

Since the gauge functions are asymptotically ordered

$$\begin{aligned} a_1 &= \lim_{\epsilon \to 0} \frac{u(\epsilon)}{\phi_1(\epsilon)} \\ a_k &= \lim_{\epsilon \to 0} \frac{u(\epsilon) - \sum_{i=0}^{k-1} a_i \phi_i(\epsilon)}{\phi_k(\epsilon)} \end{aligned}$$

Example 6. *First we note the asymptotic ordering*

$$\cos \epsilon \ll \sin \epsilon \ll \epsilon^2$$

and seek to represent

$$\sqrt{4+\epsilon} = a_1 \phi_1 + a_2 \phi_2 + a_3 \phi_3 + o(\phi_3)$$

We use the sequence of calculations

$$\begin{aligned} a_1 &= \lim_{\epsilon \to 0} \frac{\sqrt{4+\epsilon}}{\cos \epsilon} = 2 \\ a_2 &= \lim_{\epsilon \to 0} \frac{\sqrt{4+\epsilon} - 2\cos\epsilon}{\sin \epsilon} = \frac{1}{4} \\ a_3 &= \lim_{\epsilon \to 0} \frac{\sqrt{4+\epsilon} - 2\cos\epsilon - \frac{1}{4}\sin(\epsilon)}{\epsilon^2} = \frac{63}{64} \end{aligned}$$

to conclude

$$\sqrt{4+\epsilon} = 2\cos(\epsilon) + \frac{1}{4}\sin(\epsilon) + \frac{63}{64}\epsilon^2 + o(\epsilon^2)$$

1.2 Uniformity

When asymptotic expansions are used to approximate solutions of differential equations having a small parameter ϵ, the unknown function $u(x,\epsilon)$ depends on x and ϵ. More generally, let us consider some domain $\Omega \subset \mathbb{R}^n$ and $u, \phi : \Omega \times \bar{N}_r(0) \to \mathbb{R}$. Then the truth of the statement

$$u(x,\epsilon) \sim \phi(x,\epsilon) \qquad \epsilon \to 0$$

will in general depend on x. If it does not then the relation is said to hold uniformly on Ω. A couple of distinguishing definitions are:

Definition 8. *$u = O(\phi)$ as $\epsilon \to 0$ if $\exists k(x)$ independent of ϵ and defined on Ω such that*

$$|u(x,\epsilon)| \leq k(x)\ |\phi(x,\epsilon)| \qquad\qquad \forall x \in \Omega$$

for all ϵ in some $\bar{N}_r(0)$.

Definition 9. *$u = O(\phi)$ uniformly on Ω if there exists constants M, r independent of ϵ such that*

$$|u(x,\epsilon)| \leq M\ |\phi(x,\epsilon)| \qquad\qquad \forall x \in \Omega$$

for all $\epsilon \in \bar{N}_r(0)$.

Example 7. *Let $\Omega = (0,1)$ and $\epsilon \in (0,1)$, then*

$$x + \epsilon = O(1) \qquad uniformly \;\; on \;\; \Omega$$

since $|x+\epsilon| \leq M = 2$ is independent on x, in particular.

Example 8. *Again let $\Omega = (0,1)$ and $\epsilon \in (0,1)$. While it is true*

$$\frac{1}{x+\epsilon} = O(1)$$

since

$$\left|\frac{1}{x+\epsilon}\right| < \frac{1}{x} = k(x)$$

it is not uniformly $O(1)$ on Ω. Were it, there would be a constant M independent of x, ϵ such that

$$\left| \frac{1}{x+\epsilon} \right| < M$$

for all $x \in \Omega$ and ϵ sufficiently small.

A couple of useful Theorems [1] related to uniformity we list below. These can be generalized but in both $\epsilon \in I = (0, r)$, $\Omega = (a, b)$ and its closure $\bar{\Omega} = [a, b]$.

Theorem 6. *Let u, ϕ be continuous on $\bar{\Omega} \times I$. If*

i) $u \sim \phi$ for all $x \in \bar{\Omega}$

ii) $\tilde{k} = |\phi(x, \epsilon)|$ is a strictly decreasing function of ϵ for all $x \in \Omega$.

then $u \sim \phi$ uniformly on $\bar{\Omega}$.

Nonuniformity can be shown by the contrapositive of

Theorem 7. *Let u, ϕ be continuous on $\bar{\Omega} \times I$. If*

i) $u \sim \phi$ uniformly on $\bar{\Omega}$

ii) ϕ is bounded on $\bar{\Omega} \times I$

then

$$\lim_{\epsilon \to 0^+} \lim_{x \to x_0} u(x, \epsilon) = \lim_{\epsilon \to 0^+} \lim_{x \to x_0} \phi(x, \epsilon)$$

for all $x_0 \in \Omega$.

As an example of the usage of the latter let $u(x, \epsilon) = x + \exp(-x/\epsilon)$ and $\phi(x, \epsilon) = x$ where here $\Omega = (0, 1)$. Then

$$\begin{aligned} \lim_{\epsilon \to 0^+} \lim_{x \to 0} u(x, \epsilon) &= 1 \\ \lim_{\epsilon \to 0^+} \lim_{x \to 0} \phi(x, \epsilon) &= 0 \end{aligned}$$

Since these limits differ $u \sim \phi$ is not uniformly valid on $[0, 1]$.

[1] Introduction to Perturbation Methods, M. Holmes, 1995. pg 38-41.

Example 9. *We consider the asymptotic validity of* <u>*outer*</u>*,* <u>*inner*</u> *and* <u>*composite*</u> *expansions of*

$$u(x,\epsilon) = \exp(-x/\epsilon) + x + \epsilon \qquad x \in [0,1]$$

An <u>*outer*</u> *expansion is found by fixing x and expanding in ϵ in some set of gauge functions. Here, for fixed x,*

$$u(x,\epsilon) \;\sim\; u_0(x) \equiv x$$

Moreover, for every $\delta > 0$ there is an r such that

$$|u(x,\epsilon) - u_0(x)| < \delta \quad , \quad \epsilon \in (0,r)$$

uniformly on $[a,1]$ for any $a > 0$. Said another way

$$u(x,\epsilon) \sim u_0(x) + o(1)$$

uniformly on $[a,1]$.

We note that $u_0(x)$ is a very poor approximation at $x = 0$. In fact $u(0,\epsilon) = 1$ while $u_0(0) = 0$. To find a better asymptotic approximation near the "layer" at $x = 0$ we first let

$$U(X,\epsilon) = u(x,\epsilon) \qquad , \qquad X = \frac{x}{\epsilon}$$

Then, the leading <u>*inner*</u> *approximation (in X) is:*

$$U(X,\epsilon) = \exp(-X) + \epsilon(X+1) \;\sim\; U_0(X) \equiv \exp(-X)$$

noting that $U : \Omega^* \times I \to \mathbb{R}$ where $I = (0,r)$ and $\Omega^* = \left[0, \frac{1}{\epsilon}\right]$. Clearly, $U \sim U_0$ uniformly for $X \in [0,b]$ where b is any ϵ independent constant with $b < \epsilon^{-1}$.

The outer approximation is uniformly valid on an interval $[a,1]$ containing the right end point while the inner approximation is uniformally valid on an interval $[0,b]$ containing the left end point. Through a process called matching one can often construct an approximation uniformly valid on the entire interval. Here the <u>*composite*</u> *solution*

$$u_c(x,\epsilon) = \exp(-x/\epsilon) + x$$

is uniformly valid on the entire interval $[0,1]$ since, in particular,

$$|u(x,\epsilon) - u_c(x,\epsilon)| = \epsilon$$

does not depend on x.

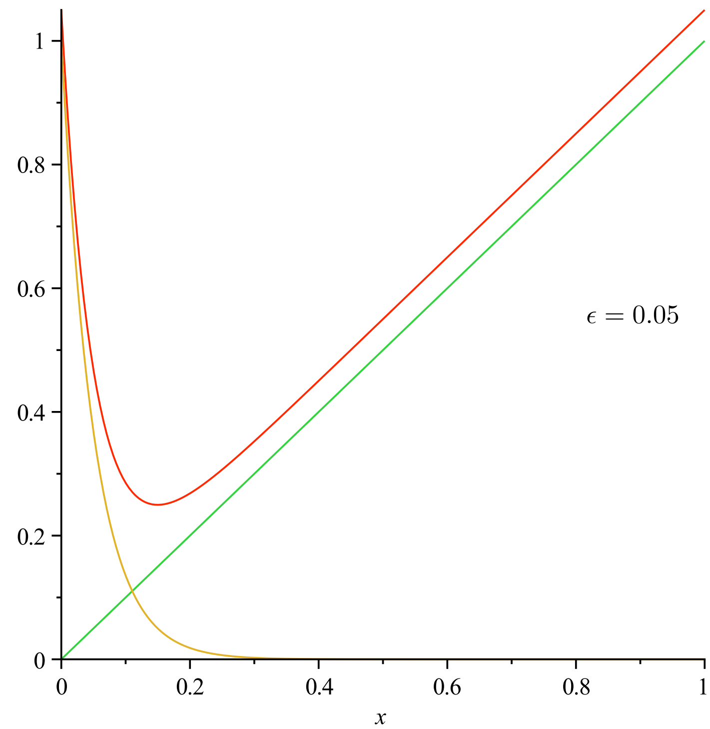

Figure 1: Inner and outer approximations of $u(x,\epsilon) = \exp(-x/\epsilon) + x + \epsilon$. The error in the outer solution (green) for $x > 0.2$ is almost constant and $O(\epsilon)$ while the inner solution (yellow) is a better approximation near $x = 0$.

Example 10. *Another similar example:*

$$u(x,\epsilon) = e^{-\frac{x}{\epsilon}} - \frac{\epsilon e^{-x}}{x+\epsilon}$$

Using the binomial theorem we get the following two term outer expansion:

$$u(x,\epsilon) \sim -\epsilon\,\frac{e^{-x}}{x} + \epsilon^2\,\frac{e^{-x}}{x^2} + O(\epsilon^3)$$

noting $e^{-\frac{x}{\epsilon}}$ *is transcendentally small so will not be part of any outer expansion. Again letting*

$$U(X,\epsilon) = u(x,\epsilon) \qquad , \qquad X = \frac{x}{\epsilon}$$

we find a two term inner expansion

$$U(X,\epsilon) \sim e^{-X} - \frac{1}{1+X} + \frac{\epsilon X}{1+X} + O(\epsilon^2)$$

If we let $\bar{u}_2(X,\epsilon)$ *be the two term outer expansion re-expressed in terms of* X*:*

$$\bar{u}_2(X,\epsilon) = -\frac{e^{-\epsilon X}}{X} + \frac{e^{-\epsilon X}}{X^2}$$

One can show the small X expansion of $\bar{u}_2$ is

$$\bar{u}_2(X,\epsilon) \sim \left(-\frac{1}{X}+\frac{1}{X^2}\right)+\epsilon\left(1-\frac{1}{X}\right)+O(\epsilon^2)$$

Next consider the two term inner approximation

$$\bar{U}_2(X,\epsilon) = e^{-X} - \frac{1}{1+X} + \frac{\epsilon X}{1+X}$$

Then for ϵ fixed, a large X expansion of $\bar{U}_2$ is

$$\bar{U}_2(X,\epsilon) \sim \left(-\frac{1}{X}+\frac{1}{X^2}\right)+\epsilon\left(1-\frac{1}{X}+\frac{1}{X^2}\right)+O\left(\frac{1}{X^3}\right)$$

Note that to $O(1)$ these expansions "match". At $O(\epsilon)$ most terms match. One might have expected this since both $\bar{u}_2$ and $\bar{U}_2$ are (different) asymptotic approximations of the same function $u(x,\epsilon)$. Later we exploit this fact to obtain composite solutions in a procedure called matching.

1.3 Invariant operations

Certain term by term operations on asymptotic series retain the ordering of the terms. For example, let $k(x) \neq 0$ be bounded on a bounded set Ω and

$$u(x,\epsilon) = u_0(x) + u_1(x)\phi_1(\epsilon) + o(\phi_1)$$

Then

$$k(x)u(x,\epsilon) = k(x)u_0(x) + k(x)u_1(x)\phi_1(\epsilon) + o(\phi_1)$$

In some instances one can multiply asymptotic expansions out term by term - to a certain order. As an example of what might go wrong consider the following two expansions

$$\begin{aligned} F(\epsilon) &= \frac{1}{1-\epsilon} \sim 1+O(\epsilon) \\ G(\epsilon) &= \sqrt{1+\epsilon} \sim 1+\frac{1}{2}\epsilon+O(\epsilon^2) \end{aligned}$$

One might carelessly and falsely conclude through multiplication of all the terms that

$$F(\epsilon)G(\epsilon) \sim 1+\frac{1}{2}\epsilon+O(\epsilon^2)$$

when in fact

$$F(\epsilon)G(\epsilon) \ \sim \ 1 + \frac{3}{2}\epsilon + O(\epsilon^2)$$

A as separate issue, one can prove that asymptotic ordering is invariant with respect to term by term integration

$$\int_\Omega u(x,\epsilon)\ dx = \int_\Omega u_0(x)\ dx + \int_\Omega u_1(x)\ dx\ \phi_1(\epsilon) + o(\phi_1)$$

Generally speaking, though, differentiation does not preserve asymptotic ordering.

Example 11. *Let the gauge functions be $\phi_n = \epsilon^n$, $n = 0, 1, 2, \ldots$ and $u(x,\epsilon) = e^{-x/\epsilon}\sin(e^{x/\epsilon})$. Relative to this asymptotic basis, to all orders*

$$u \sim 0$$

However,

$$\frac{du}{dx} = \underbrace{-\frac{1}{\epsilon}e^{-x/\epsilon}\sin(e^{x/\epsilon})}_{T.S.T.} + \underbrace{\frac{1}{\epsilon}\cos(e^{x/\epsilon})}_{O(\epsilon^{-1})}$$

So $u'(x)$ has no expansion relative to the basis $\{\phi_n\}$

Example 12. *Well ordering need not preserved when expansions are differentiated. To illustrate this let*

$$\begin{aligned} \phi_1(x,\epsilon) &= x \\ \phi_2(x,\epsilon) &= 4\epsilon\cos\left(\frac{x}{\epsilon}\right) \end{aligned}$$

Formally, $\phi_2 \ll \phi_1$ for all $x \neq 0$. However

$$\begin{aligned} \frac{d\phi_1}{dx} &= 1 \\ \frac{d\phi_2}{dx} &= 4\sin\left(\frac{x}{\epsilon}\right) \end{aligned}$$

are of the same order [2]. In fact for many x we have $\phi_2 > \phi_1$

[2] Big O sense

2 Regularity Theorems

We will let $\bar{R} = R \times N_r(0)$ where R is an open subset of $\mathbb{R}^n$. The following theorem can be used to prove roots of systems have regular expansions in ϵ.

Theorem 8. *Implicit Function Theorem*[3]*: Let* $f : \bar{R} \to \mathbb{R}^n$ *with*

$$\begin{aligned} f &\in C^k(\bar{R}) \\ f(x_0, 0) &= 0 \\ \det Df(x_0, 0) &\neq 0 \end{aligned}$$

Then there exists a function $\bar{x}(\epsilon) \in C^k(I)$ *for some* $I \subset N_r(0)$ *and containing* 0 *such that* $f(\bar{x}(\epsilon), \epsilon) = 0$ *for all* $\epsilon \in I$*.*

[3] Munkres, J. R. Analysis on Manifolds. Reading, MA: Addison-Wesley, 1991.