Asymptotic expansions for Algebraic Eqns

$$f(x, \varepsilon) = 0$$

Regular solutions are found using

$$x(\varepsilon) \sim x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

For other roots some common expansions are:

$$x(\varepsilon) = \varepsilon^{\alpha}(x_0 + \varepsilon^{\beta} x_1 + \cdots) \qquad \beta > 0,\ x_0 \neq 0$$

$$x(\varepsilon) = \mu(\varepsilon)(x_0 + \gamma(\varepsilon) x_1 + \cdots) \qquad x_0 \neq 0,\ \gamma \ll 1$$

$$x(\varepsilon) = x_0 \phi_0(\varepsilon) + x_1 \phi_1(\varepsilon) + \cdots \qquad \phi_{n+1} \ll \phi_n$$

For an assumed expansions one tries to only satisfied the equation asymptotically

$$f(x(\varepsilon), \varepsilon) \sim 0$$

to as high an order as desired.

Why such generalizations? Consider two simple examples

$$\varepsilon x^2 - 1 = 0 \qquad x = O(\varepsilon^{-\frac{1}{2}}) \quad \alpha = -\tfrac{1}{2}$$

$$\varepsilon x^3 - 1 = 0 \qquad x = O(\varepsilon^{-\frac{1}{3}}) \quad \alpha = -\tfrac{1}{3}$$

EXAMPLE Find an asymptotic expansion for the largest root of

(1) $$\varepsilon^2 x^3 - x + \varepsilon = 0$$

For polynomials a scaling method works

(2) $$x = \varepsilon^{-\alpha} X \qquad \alpha > 0$$

Substitute (2) into (1) to get

(3) $$X^3 - \varepsilon^{2\alpha-2} X + \varepsilon^{3\alpha-1} = 0$$

$$① \sim ② \gg ③$$

The indicated asymptotic balance and ordering is true if

$$\boxed{\alpha = 1}$$ "dominant balance"

which then yields for $\delta(\varepsilon) = \varepsilon^2 \ll 1$

(4) $$X^3 - X + \delta = 0$$

This is exact and has a regular expansion in δ by IFThm.

$$X = X_0 + \delta X_1 + O(\delta^2)$$

$O(1)$ $\quad X_0^3 - X_0 = 0$

$O(\delta)$ $\quad (1 - 3X_0^2)\, X_1 = 1$

The largest X_0 value is $X_0 = +1$. Hence $X_1 = -\frac{1}{2}$

$$X = 1 - \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^4)$$

and

$$x = \frac{1}{\varepsilon}\left(1 - \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^4)\right)$$

<u>Remarks</u> In this case other choices of α were possible. Consider

$$\underset{①}{X^3} - \underset{②}{\varepsilon^{2\alpha-1} X} + \underset{③}{\varepsilon^{3\alpha-1}} = 0$$

If ② ~ ③ then $\alpha = -1$, $x = \varepsilon X$ and

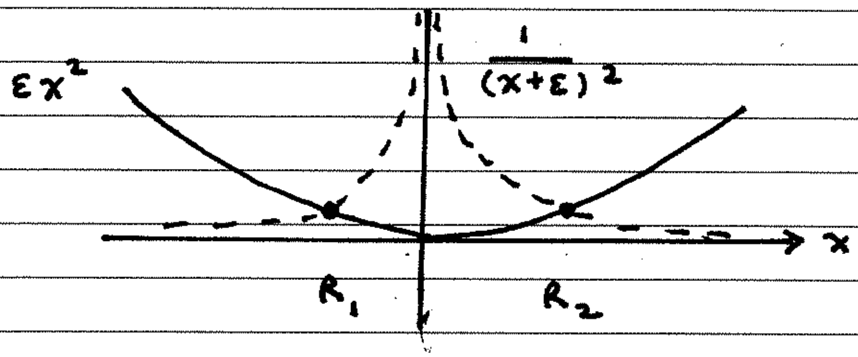
$$\varepsilon^3 X^3 - X + 1 = 0$$

A regular expansion in $\delta = \varepsilon^3$ yields a regular expansion of the small root.

If ① ~ ③ then $\alpha = \frac{1}{3}$ and ② $\gg$ ① ~ ③.
The leading value of X is zero and no root expansion is obtained under such assumptions.

EXAMPLE $\boxed{\varepsilon x^2 - \dfrac{1}{(x+\varepsilon)^2} = 0}$ $\varepsilon > 0$

This is equivalent to finding the roots of a fourth order polynomial but we won't treat it as such.



The graph indicates two real roots for $\varepsilon > 0$.

$$x = \varepsilon^{-\alpha} X \qquad \alpha > 0$$

yields

$$(1) \qquad \varepsilon^{1-2\alpha} X^2 - \frac{\varepsilon^{2\alpha}}{(X + \varepsilon^{1+\alpha})^2} = 0$$

$$① \sim ②$$

Must have ① ~ ② for $X \neq 0$. Hence

$$1 - 2\alpha = 2\alpha$$

or

$$\boxed{\alpha = \frac{1}{4}} \qquad \text{dom. balance.}$$

For $\alpha = \frac{1}{4}$ equation (1) is (exactly)

$$(2) \qquad X^2 - \frac{1}{(X+\delta)^2} = 0 \quad , \qquad \delta \equiv \varepsilon^{5/4}$$

Conditions for IVT satisfied $\Rightarrow$

$$X = X_0 + \delta X_1 + O(\delta^2)$$

Using (2)

$$O(1) \qquad X_0^2 - \frac{1}{X_0^2} = 0$$

$$O(\delta) \qquad (X_0^4 + 1) X_1 = -1$$

The roots of the $O(1)$ problem are

$$X_0 = \pm 1, \ \pm i$$

but regardless of which root

$$X_1 = -\tfrac{1}{2}$$

<u>Largest real root</u> is, therefore,

$$x = \frac{1}{\varepsilon^{1/4}} \left(1 - \tfrac{1}{2} \varepsilon^{5/4} + O(\varepsilon^{5/2}) \right)$$

$$x = \frac{1}{\varepsilon^{1/4}} - \tfrac{1}{2}\varepsilon + O(\varepsilon^{9/4})$$

EXAMPLE $\boxed{f(x,\varepsilon) = x^2 - 2\varepsilon x - \varepsilon = 0}$

A simple problem which illustrates pitfalls. Suppose we naively <u>assume</u> the small root has the expansion

(1) $$x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

Then

$$f(x,\varepsilon) = \underbrace{x_0^2} + \underbrace{(2x_0x_1 - 2x_0 - 1)}\varepsilon + O(\varepsilon^2)$$

The indicated terms must vanish but if $x_0 = 0$ the second term is $-1 \neq 0$.

The expansion (1) was an "assumption". The IVThm fails here since

$$f_x(0,0) = 0$$

so no such expansion is guaranteed.

To capture the small root behavior:

$$x = \varepsilon^{\beta} z \qquad \beta > 0$$

yields

(2) $$\underset{①}{z^2} - \underset{②}{2\varepsilon^{1-\beta} z} - \underset{③}{\varepsilon^{-2\beta+1}} = 0$$

There are three potential asymptotic balances

② ~ ③ $\Rightarrow \beta = 0$ (no scaling)

① ~ ② $\Rightarrow \beta = 1$ then ③ >> ① $\Rightarrow -1 = 0$!!

① ~ ③ $\Rightarrow \beta = \frac{1}{2}$

For the latter choice

(3) $$z^2 - 2\delta z - 1 = 0 \qquad \delta = \sqrt{\varepsilon}$$

The usual expansion

$$z = z_0 + \delta z_1 + O(\delta^2)$$

applies since $F_z(1,0) \neq 0$ and IVThm applies.

$O(1)$ $$z_0^2 - 1 = 0$$

$O(\delta)$ $$z_1 = 1$$

Hence

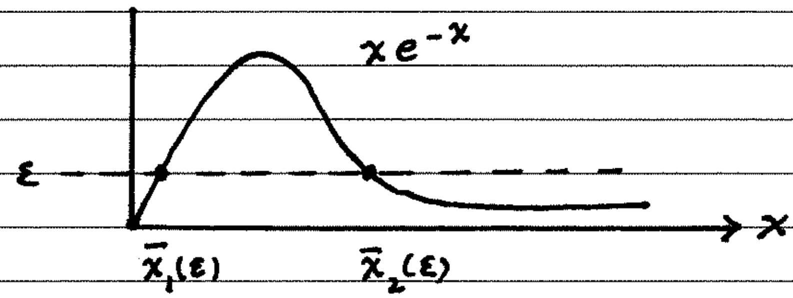
$$x = \sqrt{\varepsilon}\,(\pm 1 + \sqrt{\varepsilon} + O(\varepsilon)$$

$$x = \pm \varepsilon^{1/2} + \varepsilon + O(\varepsilon^{3/2})$$

EXAMPLE Seek asymptotic approximations to

$$f(x,\varepsilon) = xe^{-x} - \varepsilon = 0$$

The following graph indicates there are two positive roots



Using standard procedures the smaller root can be shown to have a regular expansion

$$\bar{x}_1(\varepsilon) = \varepsilon + \varepsilon^2 + \tfrac{3}{2}\varepsilon^3 + O(\varepsilon^4)$$

<u>Larger root $\bar{x}_2(\varepsilon) \gg 1$</u>

First note $f(x,\varepsilon) = 0 \Rightarrow f(x,\varepsilon) \sim 0$ so we shall seek an expansion

(1) $$\bar{x}_2(\varepsilon) \sim x_0\,\mu(\varepsilon) + o(\mu) \qquad x_0 \neq 0$$

where $x_0, \mu(\varepsilon)$ are to be determined.

Note now $f(x,\varepsilon) = 0$ is equivant to

(2) $$F(x,\varepsilon) = \log x - x - \log\varepsilon = 0$$

Substitute (1) into (2)

$$(3) \qquad \boxed{\log\big(\underset{①}{x_0\mu} + o(\mu)\big) - \underset{②}{x_0\mu} - \underset{③}{\log\varepsilon} = o(\mu)}$$

Here $\mu \gg 1$.

Must choose x_0, μ s.t. terms balance while neglected terms are smaller order.

<u>Possibility ① ~ ② can't happen</u>

wlog $x_0 = 1$ else $\mu \to x_0\mu$ in following limit:

$$\frac{\log(\mu + \phi(\mu))}{\mu} = \frac{\log(\mu[1+\frac{\phi}{\mu}])}{\mu}$$

$$= \frac{\log\mu}{\mu} + \frac{\log(1+\overset{0}{\cancel{\frac{\phi}{\mu}}})}{\mu} \qquad \phi \ll \mu$$

$$\longrightarrow 0 \quad \text{as } \mu \to \infty$$

Consequently ① ≪ ② for all $\mu \gg 1$.

<u>Possibility ① ~ ③ reproduces regular root</u>

Easy to see this if $x_0 = 1$, $\mu(\varepsilon) = \varepsilon$ but then

$$\bar{x}_2(\varepsilon) = \varepsilon + o(\varepsilon)$$

is reproducing $\bar{x}_1$.

Claim ② ~ ③ $\qquad x_0 = -1\ ,\ \mu(\varepsilon) = \ln\varepsilon$

For these choices ② = ③. The only remaining question is if the remaining term is of a smaller order:

$$\log\left(|\log\varepsilon| + o(\mu)\right) \ll \mu$$

This is easily verified so we may conclude

$$\boxed{\bar{x}_2(\varepsilon) \sim -\ln\varepsilon + o(\ln\varepsilon)}$$

Higher order correction (Bootstrap)

$$\bar{x}_2(\varepsilon) \sim -\ln\varepsilon + x_1\mu(\varepsilon) + o(\mu)$$

Find x_1 and a new $\mu(\varepsilon)$ s.t. (3) true asymptotically

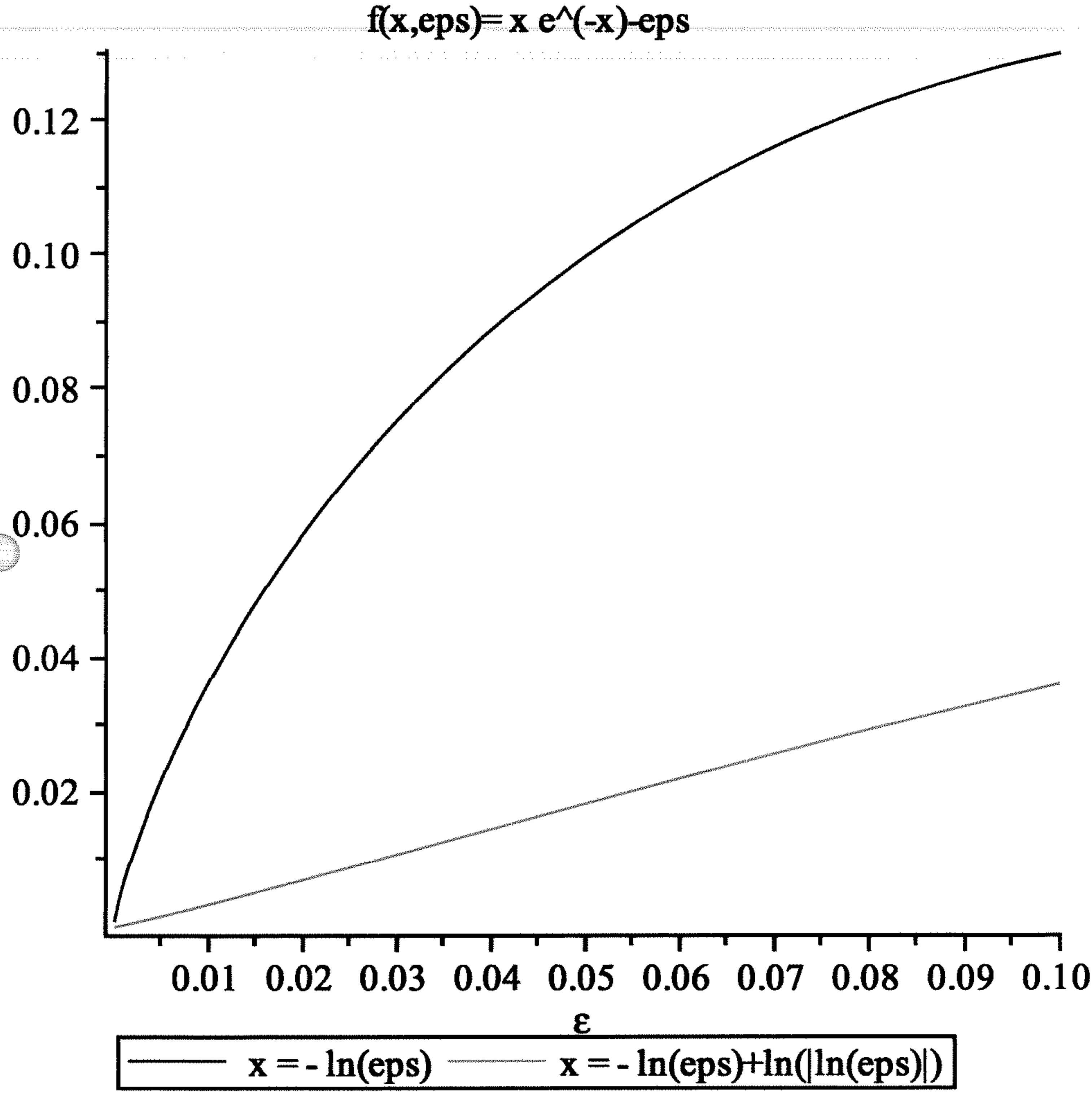
$$\log\left(-\ln\varepsilon + x_1\mu + o(\mu)\right) - x_1\mu = o(\mu)$$

Only two terms to balance

$$x_1 = +1 \qquad \mu(\varepsilon) = \log|\log\varepsilon|$$

Hence

$$\boxed{\bar{x}_2(\varepsilon) \sim |\ln\varepsilon| + \ln|\ln\varepsilon| + o(\ln|\ln\varepsilon|)}$$


f(x,eps)= x e^(-x)-eps
0.12
0.10
0.08
0.06
0.04
0.02
0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.10
ε
x = - ln(eps)
x = - ln(eps)+ln(|ln(eps)|)