FREDHOLM ALTERNATIVE (Bounded Operator/Closed Range)

If $L: H \to H$ bounded linear operator such that $R(L)$ is closed then

$$Lu = f \quad \Longleftrightarrow \quad \langle f, v \rangle = 0 \quad \forall v \in N(L^*)$$
has a solution

Proof First, suppose $v \in N(L^*) \Longleftrightarrow L^* v = 0$.

$$\langle v, f \rangle = \langle L^* v, u \rangle = 0$$

Now assume $Lu = f$ has no solution but that $\langle v, f \rangle = 0$ for all $v \in N(L^*)$.

Since $f \in R(L)$ and $R(L)$ is closed

used $R(L)$ closed

$$H = R(L) \oplus R(L)^\perp$$

Choose $f_R \in R(L)$ and $f_R^\perp \in R(L)^\perp$ so that

$$f = f_R + f_R^\perp$$

Then $\langle f_R^\perp, Lz \rangle = 0 \quad \forall z \in H$

used L bounded here.

$$\langle L^* f_R^\perp, z \rangle = 0 \quad \forall z \in H$$

Hence $f_R^\perp \in N(L^*)$. But $\langle v, f \rangle = 0 \quad \forall v \in N(L^*)$ so choose $v = f_R^\perp$

$$\langle f_R^\perp, f_R + f_R^\perp \rangle = \| f_R^\perp \|^2 = 0$$

$\Rightarrow f_R^\perp = 0$. <u>BUT</u> then $f = f_R \in R(L)$ contradicts $Lu = f$ not having a soln since $f \in R(L)$ afterall!

$\square$

EXAMPLE Consider

$$(1) \qquad u'' + \omega^2 u = f(x), \qquad u(0) = u(1) = 0$$

Solutions of (1) solve

$$(2) \qquad u(x) + \omega^2 \int_0^1 k(x,y)\, u(y)\, dy = F(x) \equiv \int_0^1 k(x,y)\, f(y)\, dy.$$

$$(3) \qquad (I + \omega^2 K)\, u(x) = F(x).$$

where

$$(4) \qquad k(x,y) = \begin{cases} y(x-1) & 0 \le y < x \le 1 \\ x(y-1) & 0 \le x < y \le 1 \end{cases}$$

and K is a bounded (continuous) operator.

Fredholm alternative states that (2) has a solution only if

$$\langle v, F \rangle = 0 \qquad \forall v \in N((I + \omega^2 K)^*)$$

$$\langle v, F \rangle = 0 \qquad \forall v \in N(I + \omega^2 K)$$

If $\omega^2 \neq \lambda_n$ where λ_n are e-vals of

$$u_n'' + \lambda_n u_n = 0 \qquad u_n(0) = u_n(1)$$

then $N(I + \omega^2 K) = \{0\}$. If $\omega^2 = \lambda_n = n^2\pi^2$ then we must have

$$\langle u_n, F \rangle = \langle \sin(n\pi x), F \rangle = 0 \qquad \square.$$

For $\lambda = 0$, $u(x) \equiv 0 \Rightarrow N(I - \lambda K)$ and alternative is trivially satisfied.

For $\lambda = 3$,

$$N(I - 3K) = \text{span}(x)$$

In this instance

$$u(x) = f(x) + 3\int_0^1 x\,t\,u(t)\,dt$$

has a soln only if

$$\langle x, f \rangle = \int_0^1 x\,f(x)\,dx = 0 \qquad \square.$$

FREDHOLM ALTERNATIVE

Let $L: H \to H$ be bounded linear operator whose range $R(L)$ is closed in H.

$$\underset{\text{has a soln}}{Lf = g} \iff \langle g, v\rangle = 0 \quad \forall v \in N(L^*)$$

EXAMPLE When does the following have a soln. $(\lambda \in \mathbb{R})$

$$u(x) = f(x) + \lambda \int_0^1 xt\, u(t)\, dt = f + \lambda K u$$

$$(I - \lambda K) u = f$$

By Fredholm alternative, has a soln $\iff$

$$\langle f, g\rangle = 0 \quad \forall g \in N((I-\lambda K)^*)$$

First note

$$\begin{aligned}\langle (I-\lambda K) f, g\rangle &= \langle f, g\rangle - \lambda \langle Kf, g\rangle \\ &= \langle f, g\rangle - \lambda \langle f, K^* g\rangle \\ &= \langle f, (I - \lambda K^*) g\rangle\end{aligned}$$

shows $(I-\lambda K)^* = I - \lambda K^*$ where $K^* = K$.

Thus, $N((I-\lambda K)^*) = N(I - \lambda K)$. Find $N(I-\lambda K)$

(1) $$u(x) = \lambda \int_0^1 xt\, u(t)\, dt = \lambda x \alpha, \quad \alpha = \int_0^1 t\, u(t)\, dt$$

Using $u(x) = \lambda x \alpha$ in $\alpha = \int_0^1 t\, u(t)\, dt$

$$\alpha = \lambda \alpha \int_0^1 t^2\, dt = \tfrac{1}{3} \lambda \alpha$$

Either $\alpha = 0$ or $\lambda = 3$.

Riesz Representation Theorem

Let $L : H \to \mathbb{R}$ be a bounded linear functional
Then $\exists!\, g \in H$ such that

$$Lf = \langle f, g \rangle \qquad \forall f \in H$$

Proof See text

Remark about proof

$$N(L) = H \qquad g = 0$$

$$N(L) \neq H \qquad g = (Lg_0)\, g_0$$

where $g_0 \in N(L)^\perp$, $\|g_0\| = 1$.

EXAMPLE Average operator on $L^2[a,b]$ has $g = \frac{1}{b-a}$.

EXAMPLE No such g for $Lu = u(0)$ on $L^2[-1,1]$

EXISTENCE OF ADJOINTS

$L : H \to H$ bounded operator then

1) L^* exists
2) L^* bounded operator

Pf/ See text.

EX **Unbounded functional:** $H = L^2[-1,1]$

$$Lu = u(0)$$

$$L : H \to \mathbb{C}$$

clearly is linear functional but not bounded.

$$u_n(x) = \begin{cases} \sqrt{n} & x \in \left(-\frac{1}{2n}, \frac{1}{2n}\right) \\ 0 & \text{otherwise.} \end{cases}$$

Have $\|u_n\| = 1$ for all n.

$$T u_n = \sqrt{n}$$

Cannot have a constant K such that

$$|Tu_n| = \sqrt{n} \leq K \|u_n\| = K \qquad \forall n.$$

Remark This functional is $\delta(x)$ in the sense that

$$\int_{-1}^{1} \delta(x) u(x)\, dx = u(0) = Lu$$

Bounded linear functionals

1) $L : H \to \mathbb{C}$

2) L linear

3) $|L(u)| \le K \|u\|$, $\forall u \in H$

<u>EX</u> $Lu = \langle u, v \rangle = \int_a^b u(x)\bar{v}(x)\,dx$, v fixed, $H = L^2[a,b]$.

$$|Lu| = |\langle u, v\rangle| \le \|u\|\,\|v\| = K\|u\| \quad , \quad \forall u \in H$$

by C-S inequality.

<u>Ex</u> $Lu = \int_0^1 u(x)\,dx$ average operator

Schwartz inequality ; $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$

$$\int_\Omega |uv|\,dx \le \left(\int_\Omega |u|^p\right)^{1/p} \left(\int_\Omega |v|^q\right)^{1/q}$$

Choose $\Omega = [0,1]$, $p = 2$, $q = 2$

$$|Lu| \le \int_0^1 |u|\,dx \le \|u\|_{L^2[0,1]}$$

shows $L : H \to \mathbb{C}$ bounded functional

However, $\nexists\, z \in H$ such that

$$Lz = y$$

If there were, by the definition of L

$$\langle z, \phi_n \rangle = 1 \qquad \forall n$$

in which case

$$z = \sum_{n \geq 1}^{\infty} \phi_n$$

But this does not converge, i.e. $z \notin H$.

<u>Remark</u> Later we will want to be able to write

$$H = N(L) \oplus N(L)^\perp$$

$$H = R(L) \oplus R(L)^\perp$$

<u>Projection Thm (Functional)</u>

If $M \subset H$ is closed then $H = M \oplus M^\perp$

<u>Range space is Linear subspace</u>

$$R(L) = \{ y \in H : \exists \ x \in D(L) \text{ s.t. } Lx = y \}$$

Let $y_1, y_2 \in R(L)$ then $Lx_1 = y_1$, $Lx_2 = y_2$ and

$$L(\alpha x_1 + \beta x_2) = \alpha y_1 + \beta y_2 \in R(L).$$

<u>Range of L bounded need not be closed</u>

Let $H = L^2[a,b]$, $L : H \to H$ be defined by

$$Lf \equiv \sum_{n=1}^{\infty} \frac{1}{n} \langle f, \phi_n \rangle \phi_n \qquad \{\phi_n\} \text{ complete orthonormal set.}$$

Clearly is defined. Bounded because Parseval $\Rightarrow$

$$\| Lf \|^2 = \sum_{n \geq 1} \frac{1}{n^2} |\langle f, \phi_n \rangle|^2 \leq \sum_{n \geq 1} |\langle f, \phi \rangle|^2 = \| f \|^2$$

Now

$$f_n \equiv \sum^{n} \phi_k \qquad f_n \in H$$

$$Lf_n = \sum^{n} \frac{1}{k} \phi_k \qquad Lf_n \in R(L).$$

$$Lf_n \to \sum^{\infty} \frac{1}{n} \phi_n = y \qquad y \in H.$$

L bounded

Defn $M \subset H$ closed linear manifold if

(1) M linear subspace

(2) $\{u_n\} \subset M$, $u_n \to u \in H \Rightarrow u \in M$

EX $N(L)$, L bounded is closed.

Pf/ Let $Lu_n = 0$, $u_n \to u \in H$.

$\|Lu\| = \|Lu - Lu_n\| \leq K \|u - u_n\| \to 0$

Thus $u \in N(L)$.

EX If L unbounded, $N(L)$ need not even be a subspace of H.

$$Lu = x\frac{du}{dx} + \tfrac{1}{2}u \qquad L : L^2[0,1] \to L^2[0,1]$$

but $Lu = 0 \Rightarrow$

$$u(x) = \frac{c}{\sqrt{x}}, \quad c \in \mathbb{R}$$

which is not in $L^2[0,1]$.

EX $M^\perp$ is closed for all subspaces $M \subset H$.

Pf/ Let $\{u_n\} \subset M^\perp$ and $u_n \to u \in H$. For all $g \in$ ..

$|\langle u, g\rangle| = |\langle u - u_n, g\rangle| \leq \|u - u_n\| \, \|g\| \to 0$

Thus $u \perp M$ and $u \in M^\perp$.