Bounded Linear Operators

Let $L: H \to H$ be a mapping, H a Hilbert space.

Defn L is linear operator if

$$L(\alpha f + \beta g) = \alpha L f + \beta L g \qquad \forall \alpha, \beta \in \mathbb{R}, \ \forall f, g \in H$$

Defn L is a bounded operator if $\exists K > 0$ s.t.

$$\|Lf\| \le K \|f\| \qquad \forall f \in H.$$

$$\boxed{\begin{array}{l}\textbf{Theorem} \quad \text{Let } L \text{ be a linear operator, } L: H \to H. \\ \qquad L \text{ continuous} \iff L \text{ bounded}\end{array}}$$

Proof First assume L is bounded. If $\|x - x_0\| \le \varepsilon/K$,

$$\|Lx - Lx_0\| = \|L(x - x_0)\| \le K \|x - x_0\| \le \varepsilon$$

shows continuity of L. Now assume L is continuous. L is continuous at 0. Then, for $\varepsilon = 1$, $\exists \delta > 0 \ni$

$$\|y\| \le \delta \quad \Rightarrow \quad \|Ly\| \le 1$$

Let $y = \beta x$, $\beta \in \mathbb{R}$. where $\beta = \delta \|x\|^{-1}$, $x \neq 0$.

$y = \delta \hat{x} = \frac{\delta x}{\|x\|}$

$$y = \delta \frac{x}{\|x\|} \qquad x \neq 0, \ x \in H.$$

Then

$$\|Ly\| = \frac{\delta}{\|x\|} \|Lx\| \le 1 \qquad \forall x \in H$$

$$(1) \qquad \|Lx\| \le \frac{1}{\delta} \|x\| \qquad \forall x \in H.$$

Also, (1) holds for $x = 0$. $\square$

EXAMPLES OF BOUNDED (CONTINUOUS) OPERATORS

EXAMPLE Identity $Lu = u$. Choose $K = 1$

EXAMPLE Let $H = L^2[0,1]$ and

$$Ku \equiv \int_0^1 k(x,y)\,u(y)\,dy$$

where we assume

$$\int_0^1\int_0^1 k^2(x,y)\,dx\,dy < \infty .$$

To show boundedness

$$\|Ku\|^2 = \int_0^1 \left(\int_0^1 k(x,y)u(y)dy\right)^2 dx$$

C-S. $|\langle f,g\rangle|^2 \le \|f\|^2\|g\|^2$

$$\le \int_0^1 \left(\int_0^1 k(x,y)^2 dy\right)\left(\int_0^1 u(y)^2 dy\right) dx$$

$$\le \|u\|^2 \int_0^1\int_0^1 k(x,y)^2\,dy\,dx .$$

EXAMPLE Let $X = BC[0,\infty)$ = bnd cont. fns on $[0,\infty)$.
Let $\|\cdot\|$ be the sup-norm

$$\|x(t)\| = \sup_{t\ge 0} |x(t)|$$

Define "average" operator

$$Tx = \frac{1}{t}\int_0^t x(s)\,ds$$

the operator is defined since by L'Hopital's rule

$$\lim_{t\to 0} \frac{1}{t}\int_0^t x(s)\,ds = x(0)$$

Also, T is bounded since

$$\left|\frac{1}{t}\int_0^t x(s)\,ds\right| \le \frac{1}{t}\int_0^t |x(s)|\,ds \le \|x\| \qquad \square$$

Theorem $L: \mathbb{X} \to \mathbb{X}$ linear. Then

$$L \text{ continuous} \iff L\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} L z_n$$

for every convergent $\{z_n\}$.

Proof $\Rightarrow$ Assume L continuous at $\bar{z}$ and let $\|z_n - \bar{z}\| \to 0$ as $n \to \infty$, $\{z_n\} \subset \mathbb{X}$.

$$\begin{aligned} \|Lz_n - L\bar{z}\| &= \|L(z_n - \bar{z})\| && \text{linearity} \\ &\le k\,\|z_n - \bar{z}\| && \text{cont}^y \\ &\to 0 \end{aligned}$$

shows

$$L\bar{z} = \lim_{n\to\infty} L z_n$$

Proof $\Leftarrow$ Naylor/Sell pg 74.

As a consequence:

Theorem $L : X \to X$ linear.

L continuous $\iff L\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \sum_{i=1}^{\infty} \alpha_i L(x_i)$

for every convergent $\{z_n\} \subset X$.

EXAMPLE Let $H = L^2[0,1]$ and

$$Lu \equiv \int_0^1 k(x,y)\, u(y)\, dy$$

Let $\{\phi_n\} \subset H$ be a basis for H.
Then, $\forall u \in H$

$$u = \lim_{N \to \infty} u_N = \lim_{N \to \infty} \sum_{n=1}^{N} a_n \phi_n$$

Since L is bounded,

$$Lu = \lim_{N \to \infty} Lu_N$$

Justifies

$$\int_0^1 k(x,y)\, u(y)\, dy = \sum_{n=0}^{\infty} a_n \int_0^1 k(x,y)\, \phi_n(y)\, dy \qquad \square$$

UNBOUNDED OPERATOR EXAMPLE

Let $H = L^2[0, 2\pi]$

$$Lu \equiv \frac{du}{dx}$$

Consider $u_n = \sin(nx)$, $\{u_n\} \subset H$. $Lu_n = n\cos(nx)$

$$\|Lu_n\|^2 = n^2 \int_0^{2\pi} \cos^2(nx)\,dx = \pi n^2$$

$$\|u_n\|^2 = \int_0^{2\pi} \sin^2(nx)\,dx = \pi$$

Then we have

$$\|Lu_n\| = n\,\|u_n\|$$

Clearly cannot have $K > 0 \ni$

$$\|Lu\| \leq K\,\|u\| \qquad \forall u \in H. \qquad \square$$

ADJOINT (OF BOUNDED) OPERATORS

The adjoint L^* (if it exists) of a (bounded) operator $L: H \to H$ is an operator for which

$$\langle Lf, g\rangle = \langle f, L^* g\rangle \qquad \forall f, g \in H.$$

EXAMPLE $Lf = Af$, $A \in \mathbb{C}^{n\times n}$, $f \in H = \mathbb{C}^n$.

$$\begin{aligned}\langle Lf, g\rangle &= (Af)^* g \\ &= f^*(A^* g) \\ &= \langle f, L^* g\rangle \qquad \forall f, g \in \mathbb{C}^n\end{aligned}$$

if we define $L^* g = A^* g$.

EXAMPLE $(Ku)(x) = \int_a^b k(x,y)\,u(y)\,dy$, $\quad K: L^2[a,b] \to L^2[a,b]$

$$\begin{aligned}\langle Ku, v\rangle &= \int_a^b \left(\int_a^b k(x,y)\,u(y)\,dy\right) v(x)\,dx \\ &= \int_a^b u(y) \left(\int_a^b k(x,y)\,v(x)\,dx\right) dy \\ &= \langle u, K^* v\rangle\end{aligned}$$

if we define $(K^* u)(x) = \int_a^b k(y,x)\,u(y)\,dy$.

For instance

$$(Ku)(x) = \int_a^b (x-2y)\,u(y)\,dy$$

$$(K^* u)(x) = \int_a^b (y-2x)\,u(y)\,dy.$$

EXAMPLE Adjoint operators need not exist.

$$Lf = \frac{df}{dx} \quad , \quad L: H \to H \,, \quad H = L^2[0,1]$$

$$\langle Lf, g \rangle = \int_0^1 f'(x) g(x)\, dx$$

$$= f(1)g(1) - f(0)g(0) + \langle f, -\frac{dg}{dx} \rangle$$

This holds for at least some $f, g \in H = L^2[0,1]$
but even for these f, g

$$\langle Lf, g \rangle = \langle f, -\frac{dg}{dx} \rangle$$

only if $f(1)g(1) - f(0)g(0) = 0$ which is certainly a restrictive class of H.

Closed Linear Manifold

A subspace $M \subset H$ is closed if $\forall \{u_n\} \subset M$, $u_n \to u \in H$, then $u \in M$.

EXAMPLE Nullspace of $L: H \to H$ bounded, linear.

$$N(L) = \{u \in H : Lu = 0\}$$

Certainly $N(L)$ is a linear subspace. Let $\{u_n\} \subset N(L)$ with $u_n \to u \in H$. Want to show $u \in N(L)$.

$$\begin{aligned} \| Lu_n - Lu \| &= \| L(u_n - u) \| \\ &\leq K \|u_n - u\| && \text{boundedness of } L \\ &\to 0 && \text{as } n \to \infty. \end{aligned}$$

This shows $Lu_n \to Lu$. But $Lu_n = 0$, $\forall n$ thus $Lu = 0 \Rightarrow u \in N(L)$. □

EXAMPLE Orthogonal Complement $M^\perp$ of $M \subset H$.

$$M^\perp = \{ f \in H : \langle f, g \rangle = 0 \quad \forall g \in H \}$$

Let $\{u_n\} \subset M^\perp$ and $u_n \to u \in H$. For each fixed $g \in M$

$$|\langle u, g \rangle| = |\langle u - u_n, g \rangle| \leq \|u - u_n\| \|g\| \to 0$$

from which we conclude $u \in M^\perp$. □

LINEAR FUNCTIONAL

1) $T : H \to \mathbb{C}$

2) $T(\alpha f + \beta g) = \alpha Tf + \beta Tg \qquad \forall \alpha, \beta \in \mathbb{C}, \ \forall f, g \in H.$

BOUNDED LINEAR FUNCTIONAL 1) - 2) AND

3) $|T(u)| \le K \|u\| \qquad \forall u \in H.$

EXAMPLE $\quad Tu = \langle u, v \rangle = \int_a^b u(x) \bar{v}(x)\, dx \quad , \quad v$ fixed

$$T : H \to \mathbb{C}$$

Clearly is linear. Boundedness:

$$|Tu| = |\langle u, v \rangle| \le \|u\| \, \|v\| \quad , \quad K = \|v\|$$

EXAMPLE $\quad Tu = u(0) \quad , \quad T : L^2[-1,1] \to \mathbb{R}.$

Is a linear functional. Not bounded as can be seen from

$$u_n(x) = \begin{cases} \sqrt{n} & x \in (-\frac{1}{2n}, \frac{1}{2n}) \\ 0 & \text{otherwise.} \end{cases}$$

Have $\|u_n\|_{L^2[-1,1]} = 1$ for all n.

$$Tu_n = u_n(0) = \sqrt{n}$$

Clearly cannot have

$$|Tu_n| = \sqrt{n} \le K$$

for any K.

Riesz Representation Theorem.

Let $T: H \to \mathbb{R}$ be a bounded linear functional. Then $\exists!\, g \in H$ such that

$$Tf = \langle f, g \rangle \qquad \forall f \in H.$$

Proof $\quad N(T) = \{ f \in H : Tf = 0 \}$

If $N(T) = H$, choose $g = 0$. wlog $N(T) \neq H$.
Choose any $g_0 \in N(T)^\perp$ with $\|g_0\| = 1$

$$g \equiv (Tg_0)\, g_0$$

$$y \equiv (Tf)\, g_0 - f(Tg_0)$$

$$Ty = 0 \quad \Rightarrow \quad y \in N(T)$$

Thus $\langle y, g_0 \rangle = 0$ by defn of g_0

$$\begin{aligned} \langle y, g_0 \rangle &= \langle (Tf) g_0 - f(Tg_0), g_0 \rangle \\ &= Tf - T(g_0) \langle f, g_0 \rangle = 0 \end{aligned}$$

Thus $\qquad Tf = \langle f, g \rangle = \langle f, (Tg_0) g_0 \rangle$

UNIQUENESS: Suppose $Tf = \langle f, g_1 \rangle = \langle f, g_2 \rangle$

$$\langle f, g_1 - g_2 \rangle = 0 \qquad \forall f \in H$$

$$\Rightarrow \quad g_1 = g_2.$$

Can be seen by letting $f = g_1 - g_2$ then

$$\|g_1 - g_2\|^2 = 0 \qquad \Box.$$

Riesz Representation Theorem Remarks

Dimensionality of $N(T)^\perp$

Either $\dim N(T)^\perp = 0, 1$ for functional $T : H \to \mathbb{R}$.
Does not depend on continuity, only linearity of T (Naylor/Sell pg 205)

Thus, the choice $g_0 \in N(T)^\perp$, $\|g_0\| = 1$ is unique up to multiplicative $\pm$, i.e. underline ~~the~~ any in proof.

Continuity of $T : H \to \mathbb{R}$

As before, if T is continuous, the nullspace $N(T)$ is closed. By the projection theorem (Naylor/Sell pg 297)

$$H = N(T) \oplus N(T)^\perp$$

where $N(T)^\perp$ is automatically closed. If $N(T) \neq H$ then $\exists\, g_0 \in N(T)^\perp$ which is nontrivial

"Continuity assures existence of $g_0 \in N(T)^\perp$, $g_0 \neq 0$ in the event $N(T) \neq H$"

EXAMPLE $T : \mathbb{R}^n \to \mathbb{R}$, $Tu \equiv u_1$, $u = (u_1, \ldots u_n)^T$

Clearly $\quad Tu = \langle u, \bar{g} \rangle \qquad \bar{g} = (1, 0, \ldots 0)^T$

Could have found by noting T linear and bounded functional since

$$|Tu| = |u_1| = \sqrt{u_1^2} \le \sqrt{u_1^2 + \cdots u_n^2} = \|u\|$$

$$N(T) = \{ u \in \mathbb{R}^n : \ u = (0, u_2, \ldots u_n)^T \}$$

$$N(T)^\perp = \{ u \in \mathbb{R}^n : \ u = (u_1, 0, \ldots 0)^T \}$$

The only $g_0 \in N(T)^\perp$ with $\|g_0\| = 1$ is $\bar{g}$

So, in proof

$$g \equiv (Tg_0)\, g_0 = g_0 = \bar{g} .$$

EXAMPLE Average operator $Tf \equiv \int_0^1 f(t)\,dt$, $T : H \to \mathbb{R}$, $H = L^2[0,1]$.

Schwartz

$$\int_\Omega |uv| \le \left(\int_\Omega |u|^p \right)^{1/p} \left(\int_\Omega |v|^q \right)^{1/q} \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1$$

Choose $v = 1$, $\Omega = [0,1]$, $p = 2$, $q = 2$

$$\int_0^1 |u(x)|\,dx \le \|u\|_{L^2[0,1]}$$

shows T bounded (linear). Clearly

$$Tf = \langle f, g \rangle$$

where $g(x) \equiv 1$.

EXAMPLE Dimensionality of $N(T), N(T)^{\perp}$.

$$T : \mathbb{R}^3 \to \mathbb{R}$$

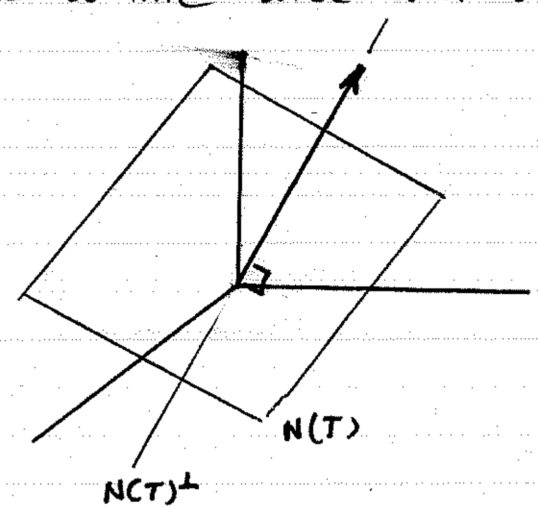
If T is bounded linear, Riesz Rep. Thm implies the existence of $\vec{a} = a \in \mathbb{R}^3$ s.t.

$$Tu = a^T u \qquad \forall u \in \mathbb{R}^3$$

The $N(T)$ is the set of $u \in \mathbb{R}^3$ s.t

$$a^T u = 0$$

which is a plane. Clearly $N(T)^{\perp}$ is a line and $\dim N(T)^{\perp} = 1$



Also, $a \perp$ plane so

$$N(T)^{\perp} = \text{span}\{a\}$$

EXISTENCE OF ADJOINTS OF BOUNDED LINEAR $L: H \to H$.

1) L^* exists
2) L^* is a bounded linear operator.

Proof Existence

$$Tu \equiv \langle Lu, v\rangle \qquad T: H \to \mathbb{C}$$

is a bounded (linear) functional since

$$|Tu| = |\langle Lu, v\rangle| \le \|Lu\| \|v\| \le K \|u\| \|v\|$$

By Riesz, $\exists!\, g$ such that

$$Tu = \langle u, g\rangle \equiv \langle u, L^*v\rangle \qquad \forall u, v \in H.$$

This defines L^*.

Linearity Let $L^*v_1 = g_1$ and $L^*v_2 = g_2$ and

$$L^*(\alpha v_1 + \beta v_2) = g_3$$

$$\begin{aligned}
\langle u, g_3\rangle &= \langle u, L^*(\alpha v_1 + \beta v_2)\rangle \\
&= \langle Lu, \alpha v_1 + \beta v_2\rangle \\
&= \bar{\alpha}\langle Lu, v_1\rangle + \bar{\beta}\langle Lu, v_2\rangle \\
&= \bar{\alpha}\langle u, g_1\rangle + \bar{\beta}\langle u, g_2\rangle \\
&= \langle u, \alpha g_1 + \beta g_2\rangle \\
&= \langle u, \alpha L^*v_1 + \beta L^*v_2\rangle
\end{aligned}$$

Boundedness

$$\|L^*v\|^2 = |\langle L^*v, L^*v\rangle| = |\langle L(L^*v), v\rangle| \le K \|L^*v\| \|v\|$$

from which $\|L^*v\| \le K\|v\|$, $\forall v \in H$. $\square$

EXAMPLE Bounded operator on $L^2[a,b]$ whose range is not closed

Let $T : H \to H$ be the bounded linear operator

$$Tf \equiv \sum_{n=1}^{\infty} \frac{1}{n} \langle f, \phi_n \rangle \phi_n$$

has countable basis

where $\{\phi_n\}$ is some orthonormal basis for H (separable) (dense count subset)

Boundedness follows from Parseval

$$\|Tf\|^2 = \sum_{n \geq 1} \frac{1}{n^2} |\langle f, \phi_n \rangle|^2 \leq \sum_{n \geq 1} |\langle f, \phi_n \rangle|^2 = \|f\|^2$$

Define

$$f_n = \sum_{k=1}^{n} \phi_k \qquad f_n \in H.$$

$$Tf_n = \sum_{k=1}^{n} \frac{1}{n} \phi_k \qquad Tf_n \in R(T)$$

Clearly

$$Tf_n \to y = \sum_{n=1}^{\infty} \frac{1}{n} \phi_n \quad \in H$$

However, $\nexists\, z \in H = L^2[a,b]$ such that $Tz = y$.
Any such z would have

$$\langle z, \phi_n \rangle = 1 \qquad \forall n$$

But

$$z = \sum_{n=1}^{\infty} \phi_n$$

does not converge. $\square$

EXAMPLE Bounded operator whose range is not closed

Let $H = \ell^2(0,\infty)$ be the (Hilbert) sequence space

(1) $x \in H \iff x = (x_1, x_2, \dots), \quad x_k \in \mathbb{C}$

(2) $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$

Now define

$$Tx = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right), \quad (Tx)_k = \frac{x_k}{k}.$$

Clearly we have linearity and boundedness by

$$\|Tx\|^2 = \sum_{n\geq 1} \frac{x_n^2}{n^2} \leq \sum_{n\geq 1} x_n^2 = \|x\|^2$$

Moreover $D(T) = H$.

Want to find z_n so that $y_n = Tz_n \to y \notin R(T)$
but for which $\nexists\, z \in D(T)$ with $Tz = y$.

$$z_n = (\overbrace{1, 1, 1, \dots 1}^{n}, 0, 0, \dots)$$

$$y_n \equiv Tz_n = (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots \tfrac{1}{n}, 0, 0, \dots) \qquad y_n \in R(T)$$

$$y \equiv (1, \tfrac{1}{2}, \tfrac{1}{3}, \dots) \qquad y \in H.$$

$$\|y_n - y\|^2 = \sum_{k=n+1}^{\infty} \frac{1}{k^2} \to 0 \quad \text{as } n \to \infty$$

shows $Tz_n \to y$. Also, $z = (1, 1, 1, \dots)$ has $Tz = y$
but $z \notin H$.

EXAMPLE Range is a subspace for L linear.

$$L : X \to Y$$

$$R(L) = \{ y \in Y : \exists x \in X,\ Lx = y \}$$

Clearly $R(L) \subset Y$. Let $y_1, y_2 \in R(L)$. Then $\exists x_1, x_2$ such that

$$Lx_1 = y_1 \qquad Lx_2 = y_2$$

The following

$$L(x_1 + x_2) = Lx_1 + Lx_2 = y_1 + y_2$$

$$L(\alpha x_1) = \alpha L x_1 = \alpha y_1 \qquad \forall \alpha \in \mathbb{C}.$$

demonstrate $R(L)$ is subspace.

Range need not be closed, however.