Fundamental Solutions.

Let $L: H \to H$ have domain $D(L)$ and assume L^* exists

By a fundamental solution for L we mean a regular distribution $g(x,t)$ that satisfies

$$Lg = \delta(x-t)$$

Fundamental solutions are not unique. For a differential operator L may have many homogeneous solns $Lg_h = 0$.

$$L(g+g_h) = \delta(x-t)$$

EXAMPLE Let $L = \nabla^2$ on $\mathbb{R}^3$ (Laplacian)

$$g = -\frac{1}{4\pi|x|}$$

Was previously shown to satisfy

(1) $$\nabla^2 g = \delta(x) \qquad x = (x_1, x_2, x_3)$$

However

$$g_h = a_1 x_1 + a_2 x_2 + a_3 x_3 \qquad a_i \in \mathbb{R}$$

so that $\tilde{g} = g + g_h$ also satisfies (1)

Greens functions

Are associate with a problem

(1) $\qquad Lu = f \qquad u \in D(L)$

which is presumed to have a unique Solution. In particular

$$N(L) \neq \phi$$

else we couldn't expect a solution

$$u = L^{-1} f = \int_{\Omega} g(x,t) f(x)\, dx$$

The integration domain depends on the problem. Some examples include

$$Lu = u' \qquad x \in [0,\infty) = \Omega$$

$$Lu = u'' \qquad x \in [a,b] = \Omega$$

$$Lu = xu'' \qquad x \in \mathbb{R} = \Omega$$

$$Lu = \Delta u \qquad x \in [0,1]^2 = \Omega$$

$$Lu = \nabla^2 u + u \qquad x \in \mathbb{R}^3$$

Domains $D(L)$ are defined by

i) boundary conditions

ii) initial conditions

Finding Greens Functions

when possible integrate by parts

(1) $\langle Lu, g\rangle = B(u,g) + \langle u, L^* g\rangle$

Try to impose conditions on $g(x,t)$ s.t.

(i) $L^* g = 0$ a.e.

(ii) $B(u,g) = u(t)$

so that if $Lu = f$ then (1) reads

$$u(t) = \langle f, g\rangle$$

Dealing with boundary terms $B(u,g)$ complex.

In other instances one tries the opposite

(i) $\langle u, L^* g\rangle = u(t)$

(ii) $B(u,g) = 0$

We show an example of the latter first.

EXAMPLE

$$\boxed{\nabla^2 u = f(x) \qquad u\big|_{\partial\Omega} = 0 \qquad \Omega \subset \mathbb{R}^3}$$

Green's 2nd identity implies

$$(1) \quad \int_\Omega g\,Lu\,dx = \int_\Omega u\,Lg\,dx + \int_{\partial\Omega} \Big(g\frac{\partial u}{\partial n} - \cancelto{0}{u\frac{\partial g}{\partial n}}\Big)\,dS(x)$$

If we choose $g = g_f$ (the fundamental soln)

$$g_f(x,x') = -\frac{1}{4\pi|x-x'|} \qquad x' \in \Omega$$

then (1) implies

$$\langle g_f, Lu\rangle = \lim_{\varepsilon\to 0}\int_{|x-x'|\geq\varepsilon} u\,Lg_f\,dx + \int_{\partial\Omega} g_f\frac{\partial u}{\partial n}\,ds'$$

$$\langle g_f, Lu\rangle = u(t) \qquad + \qquad B(u, g_f)$$

Since $B \neq 0$ we don't have desired $u(t) = \langle g_f, f\rangle$.
However we let g_h solve homog. problem

$$\nabla^2 g_h = 0 \quad , \quad g_h\big|_{\partial\Omega} = g_f\big|_{\partial\Omega}$$

Then $g \equiv g_f - g_h$ used in (1) as $\varepsilon \to 0^+$ yields

$$\langle g, Lu\rangle = u(t) + \int_{\partial\Omega} \cancelto{0}{g}\,\frac{\partial u}{\partial n}\,ds'$$

$$\langle g, f\rangle = u(t)$$

Ordinary Differential Operators.

For simplicity $a_i(x) \in C^\infty(\mathbb{R})$ and

$$Lu \equiv \sum_{i=0}^{n} a_i(x)\, u^{(i)}(x)$$

Then there is a boundary operator B_n s.t.

$$\langle Lu, v\rangle = B_n(u,v) + \langle u, L^* v\rangle$$

where the adjoint operator L^* is

$$L^* v = \sum_{i=0}^{n} (-1)^i \frac{d^i}{dx^i}(a_i v)$$

Recall $D(L^*)$ is defined by those conditions on v which assure

$$B_n(u,v) = 0 \qquad \forall u \in D(L),\ v \in D(L^*)$$

and the terminology

$L^* = L$ formally self adjoint

$L^* = L,\ D(L) = D(L^*)$ self adjoint

The boundary term depends on the order n

$$B_1(u,v) = a_1 u v$$

$$B_2(u,v) = a_2 u' v + a_1 u v - u(a_2 v)'$$

EXAMPLE First order on $\mathbb{R}^+ = [0, \infty)$

Define the inner product on $\Omega = [0, \infty)$

$$\langle u, v \rangle \equiv \int_0^\infty u(x)\, v(x)\, dx$$

Define our operator

$$Lu = -u'$$

$$D(L) = \{ u \in C^1(\Omega) : u(\infty) = 0 \}$$

Here $u(\infty) = 0$ is meant to mean $u(x) \to 0$ as $x \to \infty$ and it is implied $\|u\| < \infty$. This point of subtlety recognizes not all functions which vanish at ∞ converge in the $L^2(\mathbb{R}^+)$ sense.

(1) $$\langle Lu, g \rangle = \underbrace{-ug\Big|_0^\infty}_{B(u,g)} + \langle u, L^* g \rangle$$

Here $L^* g = +g'$. Want $B(u, g) = u(t)$ and $L^* g = 0$ a.e. Choose discontinuous $g(x,t)$

$$g(x,t) = \begin{cases} g_+(x,t) & x > t \\ g_-(x,t) & x < t \end{cases}$$

Explicitly (1) then becomes

$$\langle Lu, g\rangle = \underbrace{-ug_-\Big|_0^t - ug_+\Big|_t^{\infty}}_{\text{want } u(t)} + \underbrace{\int_0^t u L^* g_- \, dx + \int_t^{\infty} u L^* g_+ \, dx}_{\text{want to vanish}}$$

If we require $L^* g_{\pm} = 0$ and use $u(\infty) = 0$:

$$\langle Lu, g\rangle = \underbrace{u(0) g_-(0)}_{\text{want } 0} + \underbrace{u(t)\, g(x,t)\Big|_{t^-}^{t^+}}_{\text{want one.}}$$

Thus if

(2)	$L^* g_- = 0$	$g_-(0,t) = 0$	
(3)	$L^* g_+ = 0$		
(4)	$g_+(t,t) - g_-(t,t) = 1$		(Jump)

then the solution of

$$Lu = f(x) \qquad u \in D(L)$$

is

$$u(t) = \langle g, f\rangle = \int_0^{\infty} g(x,t) f(x)\, dx.$$

Ridiculously long method for integration!

Still, lets solve (2)-(4).

Since $L^* g = -g' = 0$ only if g constant then $g_\pm$ are constants

(5) $$g(x,t) = \begin{cases} A_+ & x > t \\ A_- & x < t \end{cases}$$

The boundary condition $g_-(0,t) = 0 \Rightarrow A_- = 0$.

The jump condition is then $A_+ = 1$. Then (5) $\Rightarrow$

(6) $$g(x,t) = H(x,t)$$

We recall this the distributional solution (fundamental) of

(7) $$u' = \delta(x-t)$$

Regarding $Lu = f$ and $u \in D(L)$

$$u(t) = \int_0^\infty H(x-t) f(x)\,dx$$

$$u(t) = \int_t^\infty f(x)\,dx$$

from which we readily see $u'(t) = -f(t)$.

Second Order BVP

Define the second order operator

$$Lu \equiv a_2 u'' + a_1 u' + a_0 u$$

and associated domain

$$D(L) = \{ u \in C^2[a,b] : \; B_1(u) = B_2(u) = 0 \}$$

where the separated boundary operators:

$$B_1(u) = \alpha_1 u(a) + \alpha_2 u'(a)$$

$$B_2(u) = \beta_1 u(b) + \beta_2 u'(b)$$

The solution of

$$Lu = f \qquad u \in D(L)$$

will be given by the regular distributional solution of

$$L^* g = \delta(x-t) \qquad g \in D(L^*)$$

Specifically we seek a continuous function $g(x,t)$ such that

$$u(t) = \int_a^b g(x,t) f(x)\, dx$$

where

$$g(x,t) = \begin{cases} g_+(x,t) & x > t \\ g_-(x,t) & x < t \end{cases}$$

is piecewise defined.

Seek conditions on $g_\pm(x,t)$ such that

$$\langle g, Lu \rangle = u(t)$$

Split integral into two parts.

$$\langle g, Lu\rangle = \underbrace{\int_a^t g_-(x,t)\, Lu\, dx}_{I_-} + \underbrace{\int_t^b g_+(x,t)\, Lu\, dx}_{I_+}$$

Integrating by parts twice

$$I_- = a_2 g_- u' + a_1 u g_- - u(a_2 g_-)' \Big|_a^{t^-} + \int_a^t u L^* g_-\, dx$$

$$I_+ = a_2 g_+ u' + a_1 u g_+ - u(a_2 g_+)' \Big|_{t^+}^{b} + \int_t^b u L^* g_+\, dx$$

If

$$B_1^*(g_-) = 0 \qquad B_2^*(g_+) = 0$$

where B_k^* are the adjoint boundary conditions then the $x=a$, $x=b$ boundary terms vanish for all $u \in D(L)$. Using the continuity condition on g we have

$$g_+(t^+,t) = g_-(t^-,t)$$

so the indicated terms vanish.

Thus the sum is

$$I_+ + I_- = -a_2 u \Big(\underbrace{g_-'\Big|_{x=t^-} - g_+'\Big|_{x=t^+}}_{\substack{\text{derivative} \\ \text{jump cond.}}} \Big) + \underset{\substack{\| \\ 0}}{\int_a^t u L^* g_- \, dx} + \underset{\substack{\| \\ 0}}{\int_t^b u L^* g_+ \, dx}$$

Choose $L^* g_\pm = 0$ and jumpcondition correctly and

$$\langle g, Lu \rangle = u(t).$$

<u>Summary of conditions for $g(x,t)$</u>

(1)	$L^* g_\pm = 0$	solns of homog. prob
(2)	$B_1^*(g_-) = 0$	left adjoint B.C.
(3)	$B_2^*(g_+) = 0$	right adjoint B.C.
(4)	$g_+(t^+, t) = g_-(t^-, t)$	cont^y
(5)	$\dfrac{dg}{dx}\Big\|_{t^-}^{t^+} = \dfrac{1}{a_2}$	Jump cond.

A good approach is to solve the following first

$$L^* g_+ = 0 \qquad B_2^* g_+ = 0$$

$$L^* g_- = 0 \qquad B_1^* g_- = 0$$

then consider cont^y and jump condition.

Ex Greens function for

$$u'' = f(x) \qquad u(0) = u(1) = 0$$

Problem self adjoint. Seek $g(x,t)$ s.t.

$$\frac{d^2 g}{dx^2} = \delta(x-t) \qquad g(0,t) = g(1,t) = 0$$

Find solns to

$$g_-'' = 0 \qquad g_-(0,t) = 0$$

$$g_+'' = 0 \qquad g_+(1,t) = 0$$

yields

$$g_- = A_- x \qquad g_+ = A_+(x-1)$$

Cont^y and Jump conditions

$$g\Big|_{t^-}^{t^+} = 0$$

$$g'\Big|_{t^-}^{t^+} = 1 \qquad a_2(x) = 1$$

Explicitly

$$\begin{aligned} A_- t - A_+(t-1) &= 0 \\ A_- - A_+ &= 1 \end{aligned} \qquad A_- = (1-t) \quad A_+ = -t$$

Hence

$$g(x,t) = \begin{cases} x(1-t) & x < t \\ t(1-x) & x > t \end{cases}$$

Note $g(x,t) = g(t,x)$.

__EX__ Find the Green's function for

$$u'' + u = f(x) \qquad u(0) = u'(\pi) = 0$$

Verify problem self adjoint

$$\langle Lu, v\rangle = \underbrace{(u'v - uv')\Big|_0^{\pi}} + \langle u, Lv\rangle$$

J vanishes $\iff v(0) = v'(\pi) = 0$

Hence seek solns to

$$g_-'' + g_- = 0 \qquad g_-(0,t) = 0$$
$$g_+'' + g_+ = 0 \qquad g_+'(\pi,t) = 0$$

Yields solns

$$g_- = A_- \sin x \qquad g_+ = A_+ \cos x$$

Cont^y and Jump Conditions

$$A_+ \cos t - A_- \sin t = 0$$
$$-A_+ \sin t - A_- \cos t = 1$$

Soln $A_+ = -\sin t$, $A_- = -\cos t \Rightarrow$

$$g(x,t) = \begin{cases} -\cos t \sin x & x < t \\ -\cos x \sin t & x > t \end{cases}$$

EXAMPLE Non self adjoint L

$$Lu \equiv u'' + u' \qquad x \in [0,1]$$

$$D(L) = \{u \in C^2[0,1] : u(0) = u(1) + u'(1) = 0\}$$

First we will find L^* and $D(L^*)$

$$\langle Lu, v\rangle = \underbrace{(u'v + uv - uv')}_{\text{must vanish}}\Big|_0^1 + \langle u, L^* v\rangle$$

Conclude

$$L^* v = v'' - v'$$

$$D(L^*) = \{u \in C^2[0,1] : v(0) = 0, v'(1) = 0\}$$

The Green's function for problem is

$$g(x,t) = \begin{cases} g_+(x,t) & x > t \\ g_-(x,t) & x < t \end{cases}$$

where $g(x,t)$ is continuous and

$$L^* g_- = 0 \qquad g_-(0,t) = 0$$

$$L^* g_+ = 0 \qquad g_+'(1,t) = 0$$

$$\frac{dg}{dx}\Big|_{t^-}^{t^+} = 1$$

For

$$g_- = a_0 + a_1 e^x$$

$$g_+ = b_0 + b_1 e^x$$

these conditions result in 4 eqns for a_0, a_1, b_0, b_1

$$\begin{array}{l|l} g_-(0)=0 & a_0 + a_1 = 0 \\ g_+'(1)=0 & e\, b_1 = 0 \\ \text{cont}^y & -a_0 - e^t a_1 + b_0 + e^t b_1 = 0 \\ \text{jump} & -a_1 e^t + e^t b_1 = 1 \end{array}$$

whose solution is

$$a_0 = e^{-t} \qquad a_1 = -e^{-t} \qquad b_0 = (e^{-t} - 1) \qquad b_1 = 0$$

Summary

$$g(x,t) = \begin{cases} e^{-t}(1 - e^x) & x < t \\ e^{-t} - 1 & x > t \end{cases}$$

Noteably $g(x,t)$ is not symmetric.

Sturm Liouville Self Adjoint Form

$$(1) \qquad a_2 u'' + a_1 u' + a_0 u = f(x)$$

If $a_2(x) \neq 0$ this problem can be converted to a self adjoint form

$$(2) \qquad u'' + \frac{a_1}{a_2} u' + \frac{a_0}{a_2} u = \frac{f(x)}{a_2}$$

Define the integrating factor

$$p(x) = + \exp\left(\int^x \frac{a_1(t)}{a_2(t)} \, dt \right) > 0$$

Multiply (2) thru by $p(x)$ yields

$$\boxed{Lu \equiv -(pu')' + qu = F}$$

where

$$q(x) = -\frac{a_0(x)}{a_2(x)} p(x)$$

$$F(x) = -\frac{f(x) p(x)}{a_2(x)}$$

The operator so defined is formally self adjoint

$$L\phi = L^* \phi \qquad \forall \phi \in D.$$

Green's Function for SLP

$$Lu = -(pu')' + qu = f(x) \qquad x \in [a,b]$$

$$D(L) = \{ u \in C^2[a,b] : \ B_1(u) = B_2(u) = 0 \}$$

has solution

$$u(t) = \int_a^b g(x,t) f(x)\, dx$$

where

$$(1) \qquad g(x,t) = \begin{cases} g_+(x,t) \equiv \dfrac{-u_1(x)u_2(t)}{p(t)W(t)} & x < t < b \\[2ex] g_-(x,t) \equiv \dfrac{-u_1(t)u_2(x)}{p(t)W(t)} & a < t < x \end{cases}$$

where $u_k(x)$ are independent solns of

$$(2) \qquad Lu_1 = 0 \qquad B_1(u_1) = 0 \qquad x = a$$

$$(3) \qquad Lu_2 = 0 \qquad B_2(u_2) = 0 \qquad x = b$$

and W is the Wronskian

$$W(x) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2$$

Proof is to verify same conditions that apply to L in general form.

EXAMPLE (Using SLP general result)

(1) $Lu = -(xu')'$ $\quad u(1) = 0 \quad u'(2) = 0$

is a self adjoint problem. Must solve

(2) $-(xu_1')' = 0 \quad u_1(1) = 0$

(3) $-(xu_2')' = 0 \quad u_2'(2) = 0$

General soln of $Lu = 0$ is $u(x) = c_1 + c_2 \ln x$.
Using this we find

$$u_1(x) = \ln x \qquad u_2(x) = 1$$

Calculate Wronskian

$$W = u_1 u_2' - u_1' u_2 = -\frac{1}{x}$$

Using general result for 2nd order BVP, $pW = -1$,

$$g(x,t) = \begin{cases} \ln x & x < t < 2 \\ \ln t & 1 < t < x \end{cases}$$

Higher order differential equations

$$Lu = \sum_{i=0}^{n} a_i(x) u^{(i)}(x) \qquad n\text{-th order}$$

with associated boundary/initial conditions $B_K(u) = 0$ for $K = 1, 2, \ldots n$ defining domain $D(L)$

To solve

$$Lu = f \qquad u \in D(L)$$

we need the Green's function satisfying

$$L^* g = \delta(x-t) \qquad g \in D(L^*)$$

in which case $u = \langle g, f \rangle$. Such a piecewise defined $g(x,t)$ satisfies (without proof)

1) $L^* g = 0 \qquad x \neq t$

2) $B_K^*(g) = 0 \qquad$ Adj B.C.

3) $g, g', \ldots g^{(n-2)}$ are continuous $\qquad$ Cont'y

4) $\left. \dfrac{d^{(n-1)} g}{dx^{n-1}} \right|_{t^-}^{t^+} = \dfrac{1}{a_n(t)} \qquad$ Jump

Such g are not unique if $N(L) \neq \phi$.

EXAMPLE $\quad Lu = -u''' \quad u, u', u'' \to 0$ as $x \to \infty$

Here $\Omega = \mathbb{R}^+$. Compute adjoint L^* from

$$\langle Lu, v\rangle = B(u,v) + \langle u, L^* v\rangle$$

where

$$B(u,v) = -u''v + u'v' - uv'' \Big|_0^\infty$$

Hence adjoint operator and B. Conds are

$$L^* v = v'''$$

$$v(0) = v'(0) = v''(0) = 0$$

Piecewise defined $g(x,t)$ satisfies

(1) $\quad g''' = 0 \qquad x \neq t$

(2) $\quad g_-(0) = g_-'(0) = g_-''(0) = 0$

(3) $\quad g, g'$ continuous at $x = t$

(4) $\quad g'' \Big|_{t^-}^{t^+} = -1$

Note that if

$$g(x,t) = \begin{cases} g_+(x,t) & x > t \\ g_-(x,t) & x < t \end{cases}$$

then (1) implies

$$g_\pm(x,t) = a_2^\pm x^2 + a_1^\pm x + a_0^\pm$$

The only function satisfying (1)-(2) is

$$g_-(x,t) \equiv 0 \qquad\qquad x<t$$

Thus the cont'y and jump conditions for $g_+ = a_2x^2 + a_1x + a_0$ become

$$\left\|\begin{array}{ll} a_2t^2 + a_1t + a_0 & = 0 \\ 2a_2t + a_1 & = 0 \\ 2a_2 & = -1 \end{array}\right\|$$

The solution of this system yields

$$a_2 = -\tfrac{1}{2} \qquad a_1 = t \qquad a_0 = -\tfrac{1}{2}t^2$$

Hence

$$g(x,t) = -\tfrac{1}{2}(x-t)^2 H(x-t)$$

where the Heavide function been included to reflect $g_- \equiv 0$.