Implicit Function Theorem

The basic idea of the Implicit Function Theorem is that if you know the solution to $f(y,x) = 0$ at some point then near that point y is a function of x if the Jacobian $D_y f$ in y is nonsingular. Moreover, the function is smooth in x. The latter fact is especially useful in legitimizing regular expansions of y in

$$f(y,\epsilon) = 0 \qquad , \qquad \epsilon \in N_r(0)$$

where $N_r(0)$ is a neighbourhood of $0 \leq \epsilon \ll 1$.

Theorem: Suppose $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is C^1 and

a) $f(0,0) = 0$

b) $D_y f(0,0)$ is invertible

Then there exists a neighbourhood $N_r(0)$ of $x = 0$ and a unique C^1 function $\bar{y}(x)$ such that

i) $\bar{y}(0) = 0$

ii) $f(\bar{y}(x), x) = 0$ for all $x \in N_r(0)$

There are various versions of the Theorem. The simplest involves $m = n = 1$ and the more complex involve x, y being elements of Banach spaces. Many proofs require the use of the Contraction Mapping Theorem. To see why this Theorem might be true let $f = (f_1, \dots f_n)$ and similar notations for $y \in \mathbb{R}^n, x \in \mathbb{R}^m$, differentiate

$$f_i(y(x), x) = 0$$

in x_k. In component form using repeated index sum notation:

$$\frac{\partial f_i}{\partial y_j}\frac{\partial y_j}{\partial x_k} + \frac{\partial f_i}{\partial x_k} = 0 \qquad , \qquad k = 1, 2, \dots m$$

For each k this a formula for the partial derivatives of y in x_k:

$$Df_y \, \frac{\partial y}{\partial x_k} + \frac{\partial f}{\partial x_k} = 0$$

where Df_y is the Jacobian of f in y. If Df is invertible, these systems can be solved and all partials of y are known. Essentially this means the first two terms of the Taylor Series of $\bar{y}$ are known and computable. In the simple $m = n = 1$ case this yields the familiar implicit differential formula

$$\frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)}$$