Compact Sets, Operator norms, Compact operators

In the following definitions and theorems, H is a Hilbert space though most definitions hold for general metric and/or normed vector spaces.

Compact Sets

Let $S \subset H$ be some set. Then

(C1) S bounded	$\Leftrightarrow$	$\exists M > 0$ s.t. $\| x \| \leq M, \forall x \in S$
(C2) S compact	$\Leftrightarrow$	Every sequence $\{x_n\} \subset S$ contains a convergent subsequence $\{x_{n_k}\}$ which converges to $x \in S$
(C3) S bounded	$\not\Rightarrow$	S compact
(C4) S (sequentially) compact	$\Rightarrow$	S closed and bounded
(C5) $S \equiv \{x \in H : \| x \| \leq 1\}$ compact	$\Rightarrow$	$dim(H) < \infty$

Definition: Bounded Operator An operator $L : H \rightarrow H$ is bounded if there exists some $M > 0$ such that

$$\| Lx \| \leq M \| x \| \quad \forall x \in H \tag{1}$$

Definition: Operator norm For any bounded operator $L : H \rightarrow H$ we define the operator norm as

$$\| L \|_{op} \equiv \max_{\|x\|=1} \| Lx \| = \max_{x \neq 0} \frac{\| Lx \|}{\| x \|} \tag{2}$$

Definition: Compact Operator An operator $K : H \rightarrow H$ is compact if it transforms bounded sets into compact sets.

Compact and Bounded linear operators

(CO1) K compact	$\Rightarrow$	K bounded
(CO2) K linear, $dim(R(K)) < \infty$	$\Rightarrow$	K compact
(CO3) K bounded, $\{\phi_n\}_{n=1}^{\infty}$ orthonormal	$\Rightarrow$	$\lim_{N \to \infty} K\phi_n = 0$
(CO4) K_n compact, $\| K_n - K \|_{op} \to 0$	$\Rightarrow$	K compact
(CO5) K compact	$\Leftrightarrow$	$\{x_n\} \subset H$ bounded $\Rightarrow$ $\{Lx_n\}$ has a convergent subsequence
(CO6) K_1, K_2 compact	$\Rightarrow$	$K_1 + K_2$ compact

Combining (CO4) and (CO6) we see the space of compact operators is a closed linear space using the operator norm.

(C3) Bounded S $\not\Rightarrow$ Compact set.

Let $S = \{\phi_1, \phi_2, \ldots \phi_n, \ldots\}$ orthonormal set.

$$\|\phi\| \leq 1 \quad \forall \phi \in S \quad \Rightarrow \quad S \text{ bounded set.}$$

But

$$\|\phi_n - \phi_m\|^2 = 2(1 - \delta_{nm})$$

implies distance between different elements is $\sqrt{2}$ $\Rightarrow$ only convergent subsequence must have $\phi_{n_k} = \phi_N$ (N-fixed) for all K.

So the sequence $\{\phi_n\}_{n=1}^{\infty}$ does not have convergent subsequence.

(C4) Compact $\Rightarrow$ Closed and Bounded.

Let $\{x_n\} \subset S$ with $x_n \to \bar{x} \in \bar{S}$ (closure). Since S compact, $x_{n_k} \to x' \in S$. Want to show $x' = \bar{x}$:

$$\|x' - \bar{x}\| \leq \underbrace{\|\bar{x} - x_n\|}_{\downarrow \, 0} + \underbrace{\|x_n - x_{n_k}\|}_{\downarrow \, \text{Cauchy}} + \underbrace{\|x_{n_k} - x'\|}_{\downarrow \, 0}$$

shows S closed.

Suppose S not bounded. Then $\exists \{x_n\} \subset S \ni$

$$\|x_n\| \geq n$$

which clearly has no convergent subsequence. Contradiction. $\square$

FREDHOLM ALTERNATIVE (Bounded op/closed range)

Let i) $L : H \to H$, linear

ii) L bounded operator

iii) $R(L)$ closed in H.

then

$Lu = f$ has a solution $\iff$ $\langle f, v\rangle = 0 \quad \forall v \in N(L^*)$

<u>Proof</u> $\Rightarrow$ Let $v \in N(L^*)$. By assumption $\exists u \ni Lu = f$.

$$\langle v, f\rangle = \langle v, Lu\rangle = \langle L^* v, u\rangle = \langle 0, u\rangle = 0.$$

$\Leftarrow$ Since $R(L)$ is closed, projection Thm implies $H = R(L) \oplus R(L)^\perp$. Let

$$f = f_R + f_R^\perp$$

Want to show $f_R^\perp = 0$. Since $Lz \in R(L)$

$$\langle f_R^\perp, Lz\rangle = 0, \quad \forall z \in H$$

$$\langle L^* f_R^\perp, z\rangle = 0, \quad \forall z \in H$$

$$f_R^\perp \in N(L^*)$$

By supposition $\langle f_R^\perp, f\rangle = 0$. But

$$\langle f_R^\perp, f\rangle = \cancelto{0}{\langle f_R^\perp, f_R\rangle} + \| f_R^\perp \|^2 = 0$$

implies $f_R = 0$. $\square$.

$N(K_0^* - \lambda_0 I)$, $A^T = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$, $K_0^* v = \lambda_0 v$

In general,

$$\lambda_0 v(x) = (K_0^* v)(x) = \int_0^1 k_0(y,x)\, v(y)\, dy$$

$$\lambda_0 v(x) = \beta_1 q_1(x) + \beta_2 q_2(x)$$

$$\vec{\beta} \in N(A^T - \lambda_0 I)$$

Choose $\vec{\beta} = (1,1)^T$ so that (e-fn) wlog

$$v(x) = 1 - x$$

(Can check $K_0 v = \frac{1}{2} v$).

Fredholm alternative implies

$$\langle v, f \rangle = \int_0^1 (\lambda_1 - 1)(1-x)\, dx = 0$$

or that $\lambda_1 = 1$ so

$$\lambda = \frac{1}{2} + \varepsilon + O(\varepsilon^2) \qquad \square$$

$\underline{EX}$ $(K_\varepsilon u)(x) \equiv \int_0^1 (1-y)\, u(y)\, dy + \varepsilon \int_0^1 u(y)\, dy$

Note: $k_0(x,y) = 1 \cdot 1 + 1\,(-y) = p_1(x) q_1(y) + p_2(x) q_2(y)$.

$$A = \begin{bmatrix} \langle q_1, p_1 \rangle & \langle q_1, p_2 \rangle \\ \langle q_2, p_1 \rangle & \langle q_2, p_2 \rangle \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

So e-values of K_0 are those of A.

$$\det(A - \lambda I) = \lambda^2 - \tfrac{1}{2}\lambda$$

So $\lambda_0 = 0, \frac{1}{2}$ are eigenvalues.

Expand $\lambda(\varepsilon) = \frac{1}{2} + \varepsilon \lambda_1 + O(\varepsilon^2)$, $\lambda_0 = \frac{1}{2}$.

$$(K_0 - \lambda_0 I) u_1 = \lambda_1 u_0 - K_1 u_0$$

where

$$u_0(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x)$$

$$\vec{\alpha} = (2, -1)^T \in N(A - \lambda_0 I)$$

wlog $u_0(x) = 1$, $\|u_0\| = 1$.

$$(K_0 - \lambda_0 I) u_1 = \lambda_1 - 1$$

$$L\, u_1 = f$$

Need $v \in N(K_0^* - \lambda_0 I) \neq \{0\}$ on account λ_0 also e-value of K_0^*.

Eigenvalues of separable kernel int. operators.

Let

$$\mathbb{K}u \equiv \int_a^b k(x,y)\,u(y)\,dy$$

$$k(x,y) = \sum_{i=1}^{n} p_i(x)\,q_i(y)$$

The eigenvalues of $\mathbb{K}$ are eigenvalues of

$$A = \begin{bmatrix} \langle q_1, p_1\rangle & \langle q_1, p_2\rangle & - & - \\ - & - & - & - \end{bmatrix}$$

; $\langle \cdot,\cdot \rangle = L^2[a,b]$ inner prod.

and the eigen fns (for $\lambda \neq 0$)

$$u_\lambda(x) = \sum_{i=1}^{n} \alpha_i\, p_i(x) \qquad \vec{\alpha} \in N(A - \lambda I).$$

For the adjoint problem

$$\mathbb{K}^* v \equiv \int_a^b k(y,x)\,v(y)\,dy$$

$$K^* v = \mu v$$

Eigenvalues μ are those of A^T with A defined above. Use $\det B = \det B^T$ as follows:

$$\det(A - \lambda I) = \det\left[(A-\lambda I)^T\right] = \det(A^T - \lambda I)$$

Since characteristic polynomials same, so are e values.

EXAMPLE $(K_\varepsilon u)(x) \equiv \int_0^1 xy\, u(y)\, dy + \varepsilon \int_0^1 u(y)\, dy.$

Find $\lambda_0 = 1/2$, $u_0(x) = 2x$.

Since $K_0 = K_0^*$,

$$N(K_0^* - \lambda_0 I) = \text{span}\{u_0\}.$$

Compute

$$\lambda_1 u_0 - K_1 u_0 = \lambda_1 u_0 - 1$$

Solvability

$$\tfrac{4}{3}\lambda_1 = \int_0^1 \lambda_1 (2x)^2 = \int_0^1 (2x)\, dx = 1$$

Thus $\lambda_1 = 3/4$.

Perturbed Evalue problems

$$(K_0 + \varepsilon K_1) u = \lambda$$

Suppose $K_0 \lambda_0 = \lambda_0 u_0$, $u_0 \neq 0$, $\lambda_0 \neq 0$. If

$$\lambda = \lambda_0 + \varepsilon \lambda_1 + \cdots$$

$$u = u_0 + \varepsilon u_1 + \cdots$$

Then

$$(K_0 - \lambda_0 I) u_0 = 0$$

$$(K_0 - \lambda_0 I) u_1 = \lambda_1 u_0 - K_1 u_0$$

Solvability condition.

$$\langle v, \lambda_1 u_0 - K_1 u_0 \rangle = 0 \qquad \forall v \in N(K_0^* - \lambda_0 I)$$

determines λ_1 via

$$\lambda_1 = \frac{\langle v, K_1 u_0 \rangle}{\langle v, u_0 \rangle}$$

FREDHOLM ALTERNATIVE (Closed Range)

Let $L: H \to H$ be bounded linear op, $R(L)$ closed in H.

1) $Lf = g$ has a soln $\iff$ $\langle g, v \rangle = 0 \quad \forall v \in N(L^*)$

2) $Lf = g$ has a unique soln $\iff$ $N(L) = \{0\}$

EXAMPLE $\quad u(x) = f(x) + \lambda \int_0^1 xt\, u(t)\, dt = f + \lambda K u$

Sum of bnd operators bnd. $N(I - \lambda K) = N(I - \lambda K^*) = \text{span}\{x\}$ for $\lambda = 3$. Solvability condition

$$\langle x, f \rangle = \int_0^1 x f(x)\, dx = 0.$$

EXAMPLE $\quad u'' + \omega^2 u = f(x), \quad u(0) = u(1) = 0$

$$k(x,y) = \begin{cases} y(x-1) & 0 \le y < x \le 1 \\ x(y-1) & 0 \le x \le y \le 1 \end{cases}$$

(BVP) has soln

$$u(x) + \omega^2 \int_0^1 k(x,y)\, u(y)\, dy = F(x) \equiv Kf$$

Solvabitity condition

$$\langle v, F \rangle = 0 \qquad \forall v \in N(I + \omega^2 K) = N(I + \omega^2 K^*)$$

If $\omega^2 \neq \lambda_n = n^2\pi^2$ has unique soln. If $\omega^2 = n^2\pi^2$ must have

$$\langle \sin(n\pi x), F \rangle = 0 \quad \blacksquare.$$

PROOF K_n compact, $\|K_n - K\|_{op} \to 0 \Rightarrow K$ compact

Must show $\{Ku_n\}$ has a convergent subsequence for $\{u_n\}$ bounded.

K_1 compact $\Rightarrow$ subsequence $\{u_n^{(1)}\}$ of $\{u_n\}$, $K_1 u_n^{(1)}$ converges

K_2 compact $\Rightarrow$ subsequence $\{u_n^{(2)}\}$ of $\{u_n^{(1)}\}$, $K_2 u_n^{(2)}$ converges

K_m compact $\Rightarrow$ subsequence $\{u_n^{(m)}\}$ of $\{u_n^{(m-1)}\}$, $K_m u_n^{(m)}$ converges

Define $\{v_n\}$ as 'diagonal'

$$v_n = u_n^{(n)}$$

Has the property $\{K_m v_n\}$ is convergent for every fixed m.

Now show $\{Kv_n\}$ is Cauchy.

$$\|Kv_n - Kv_m\| \leq \underbrace{\|Kv_n - K_j v_n\|}_{\substack{\downarrow \\ 0 \\ \text{by assumpt} \\ \|K-K_j\| \to 0}} + \underbrace{\|K_j v_n - K_j v_m\|}_{\substack{\downarrow \\ 0 \\ \text{since } \{K_j v_n\} \\ \text{Cauchy}}} + \underbrace{\|Kv_m - K_j v_m\|}_{\substack{\downarrow \\ 0 \\ \text{by assumpt} \\ \|K-K_j\| \to 0}}$$

Since $\{Kv_n\}$ is Cauchy and $K(S) \subset H$ complete, $Kv_n \to y \in H$.

(Since K is bounded; $\|Kx\| \leq \|Kx - K_n x\| + \|K_n x\| \leq M\|x\|$; it is continuous

K bounded, $\{\phi_n\}$ orthonormal $\Rightarrow$ $\lim_{n\to\infty} K\phi_n = 0$

Suppose not, then there is a subsequence $\{\psi_n\}$ such that

(1) $\qquad \|K\psi_n\| \geq \varepsilon > 0 \quad , \quad \forall n$

Since $\|\psi_n\| = 1$, $\{\psi_n\}$ is a bounded set.

Thus, the set $\{K\psi_n\}$ has a convergent subsequence due to K being compact. Let $\{f_n\}$ be that sequence such that

$$K f_n \to f \qquad \text{as } n \to \infty$$

(2) $\qquad \langle K f_n, f \rangle \to \|f\|^2 \neq 0 \qquad$ due to (1)

On the other hand f_n are orthogonal.

Bessel's inequality for orthogonde $\{f_n\}$

$$\sum_{n=1}^{\infty} \langle f_n, g\rangle^2 \leq \|g\|^2 < \infty \quad \Rightarrow \quad \langle f_n, g\rangle \to 0 \text{ as } n \to \infty$$

Using this (using K bounded $\Rightarrow$ K^* exists)

$$\langle K f_n, f\rangle = \langle f_n, K^* f\rangle = \langle f_n, g\rangle \to 0$$

But this contradicts (2). $\square$

EXAMPLE ILLUSTRATING (C3)

Let $S = \{\phi_1, \phi_2 .. \phi_n, ...\}$ be the countable collection of orthonormal fns in Hilbert space H. They need not be a basis for H.

Clearly S is bounded since $\|\phi\| = 1$, $\forall \phi \in S$.

However

$$\begin{aligned} \|\phi_n - \phi_m\|^2 &= \langle \phi_n - \phi_m, \phi_n - \phi_m \rangle \\ &= \|\phi_n\|^2 - 2\langle \phi_n, \phi_m \rangle + \|\phi_m\|^2 \\ &= 2 - 2\delta_{nm} \end{aligned}$$

Thus the distance between different elements is $\sqrt{2}$.

Any convergent subsequence chosen from $\{x_n\} \subset S$ must be Cauchy. Therefore any $\{x_n\} \subset S$ which has a convergent subsequence $\{y_n\}$ must have $y_n = \phi_N$, N fixed, $\forall n \geq N$.

Clearly, not all sequences

COMPACT SETS ARE CLOSED AND BOUNDED.

Let S be compact and choose $\{x_n\} \subset \overline{S}$ so that $x_n \to x \in \overline{S}$. Since S is compact $\exists \{x_{n_k}\} \subset S$ s.t $x_{n_k} \to x' \in S$.

$$\|x - x'\| \leq \underbrace{\|x - x_n\|}_{\downarrow 0} + \underbrace{\|x_n - x_{n_k}\|}_{\text{Cauchy}} + \underbrace{\|x_{n_k} - x'\|}_{\downarrow 0}$$

Hence $x = x'$.

__Bounded__ Suppose not then $\exists \{x_n\} \subset S$ such that

$$\|x_n\| \geq n$$

Every subsequence is unbounded and hence cannot converge since convergent sequences must be bounded. (Krey pg 77).

PROOF K COMPACT OP $\Rightarrow$ K BOUNDED OPERATOR

Define

$$S = \{ x \in H : \|x\| = 1 \} \qquad \text{unit sphere}$$

$$\|Kx\| = \|x\| \, \|K\hat{x}\| \qquad \hat{x} \in S$$

$$\leq M \|x\|$$

where $M < \infty$ since K compact $\Rightarrow$ K(S) bounded $\Rightarrow$ $\|K\hat{x}\| < M < \infty$ for some M.

REMARK K^* therefore exists

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(C5) $S \equiv \{x \in H : \| x \| \leq 1\}$ compact	$\Rightarrow$	$dim(H) < \infty$

Definition: Bounded Operator An operator $L : H \to H$ is bounded if there exists some $M > 0$ such that

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Compact and Bounded linear operators

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Combining (CO4) and (CO6) we see the space of compact operators is a closed linear space using the operator norm.

If $\mu = \frac{1}{\lambda}$ is an eigenvalue of K

$$N(I - \lambda K) = E_\mu(K) \neq \{0\}$$

Then,

$$(I - \lambda K)u = f$$

has a solution $\iff$ $\langle f, v \rangle = 0$, $\forall v \in E_\mu(K)$.
Solution is not unique then since if u is a solution then for any $v \in E_\mu(K)$

$$\begin{aligned}(I - \lambda K)(u+v) &= (I - \lambda K)u + (I + \lambda K)v \\ &= (I - \lambda K)u + 0 \\ &= f\end{aligned}$$

demonstrates $u+v$ is also a solution.

SOLUTION OF INTEGRAL EQNS WITH (COMPACT) SELF ADJOINT K.

Suppose that we know

(1) $$(I - \lambda K)u = f$$

has a soln, and K compact, self adjoint. Eqn (1) $\Leftrightarrow$

$$\lambda K u = u - f = g$$

Since $g \in R(K)$ can expand $g(x)$ in terms of (nonzero) eigenfns of $K = K^*$

(2) $$g(x) = \sum_{n=1}^{\infty} \langle g, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} g_n \phi_n(x)$$

Now since $u(x) = f(x) + g(x)$ from (1) we have

$$u - \lambda K u = f$$
$$f + g - \lambda K(f+g) = f$$
$$g - \lambda K g = \lambda K f$$

But since K is continuous and $K\phi_n = \mu_n \phi_n$

(3) $$\lambda K f = \sum_{n=1}^{\infty} g_n (1 - \lambda \mu_n) \phi_n(x)$$

Again $Kf \in R(K)$ so

$$Kf = \sum_{n=1}^{\infty} \langle Kf, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} \langle f, K^* \phi_n \rangle \phi_n$$

(4) $$Kf = \sum_{n=1}^{\infty} \mu_n \langle f, \phi_n \rangle \phi_n(x) \qquad f_n = \langle f, \phi_n \rangle$$

Using this in (3) it follows from orthogonality of ϕ_n.

$$g_n(1-\lambda\mu_n) = \lambda\mu_n \langle f, \phi_n \rangle$$

so that

$$g_n = \frac{\lambda\mu_n \langle f, \phi_n\rangle}{(1-\lambda\mu_n)}$$

But $u(x) = f(x) + g(x)$ so

$$u(x) = f(x) + \sum_{n=1}^{\infty} \frac{\lambda\mu_n \langle f, \phi_n\rangle}{(1-\lambda\mu_n)} \phi_n(x)$$

$$u(x) = f(x) + \lambda(Rf)(x) = (I+\lambda R)f$$

where

$$(Rf)(x) = \int_a^b r(x,y;\lambda)\, f(y)\, dy$$

$(I-\lambda K)^{-1}$ is resol. op

is the ~~RESOLVENT/ OPERATOR and the~~ resolvent kernel

$$r(x,y;\lambda) = \sum_{n=1}^{\infty} \frac{\mu_n \phi_n(x)\phi_n(y)}{(1-\lambda\mu_n)}$$

<u>REMARK</u> R fails to exist if $\lambda = 1/\mu_n$. If so $N(I-\lambda K) \neq \{0\}$ so soln not unique. Also, soln exists only if $\langle f, \phi_n\rangle = 0$ for case $\lambda = 1/\mu_n$. Here ϕ_n in $N((I-\lambda K)^*)$.

K compact, selfadjoint, $K: H \to R(K) \subset H$

Let $\lambda_n \in \mathbb{R}$, $K\phi_n = \lambda_n \phi_n$, $\|\phi_n\| = 1$.

$$(\lambda I - K)u = f, \qquad \lambda \in \mathbb{R}$$

$\underline{\lambda \neq \lambda_n}$ Since $R_\lambda \equiv \lambda I - K$ has closed range, selfadjoint and compact, Fredholm alternative applies. But $N(\lambda I - K)^* = \{0\}$ so $\langle f, v \rangle = 0$ $\forall v \in N(\lambda I - K)^*$ and all $f \in H$. But R_λ self adjoint, compact $\Rightarrow \{\phi_n\}$ complete over H. Let

$$u = \sum u_n \phi_n \qquad f = \sum f_n \phi_n$$

and derive

$$u(x) = R_\lambda^{-1} f = \sum_{n=1}^{\infty} \frac{f_n}{\lambda - \lambda_n} \phi_n(x)$$

$$(R_\lambda^{-1} f)(x) = \int_a^b r_\lambda(x, y; \lambda) f(y)\, dy$$

$$r_\lambda(x, y; \lambda) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda - \lambda_n}$$

Compare with $(I - \tilde{\lambda} K)u = f$

$$u(x) = f(x) + \int_a^b \underbrace{\left(\sum_{n=1}^{\infty} \frac{\lambda_n \phi_n(x)\phi_n(y)}{(1 - \tilde{\lambda}\lambda_n)} \right)}_{r(x,y;\lambda)} f(y)\, dy$$

$r(x,y;\lambda)$ resolvent kernel

R_λ = resolvent operator.

$\underline{\lambda = \lambda_n}$ $\quad (\lambda_n I - K)u = f$ has no solution
if

$$\langle f, v \rangle \neq 0 \qquad \text{for some } v \in E_{\lambda_n}(K)$$

$\underline{\lambda = \lambda_n}$ $\quad (\lambda_n I - K)u = f$ has a (nonunique) solution
if

$$\langle f, v \rangle = 0 \qquad \forall v \in E_{\lambda_n}(K)$$

If $f \in H$ in this case, let

$$f_\perp \equiv f - \sum_{K=1}^{N} \langle f, \phi_K \rangle \phi_K$$

where $E_{\lambda_n}(K) = \text{span}\{\phi_1, \ldots \phi_N\}$. Then

$$(\lambda_n I - K)u = f_\perp$$

$$u(x) = f(x) + \int_a^b \underbrace{\left(\sum_{\lambda_j \neq \lambda_n} \frac{\lambda_j \; \phi_j(x)\phi_j(y)}{\lambda_n - \lambda_j} - \sum_{\lambda_j = \lambda_n} \phi_j(x)\phi_j(y) \right)}_{r_\perp(x, y, \lambda_n)} f(y)\,dy$$

Psedo-resolvent kernel.

Pseudo resolvents

Let $\lambda_K = \frac{1}{\mu_K}$ where μ_K is an eigenvalue of selfadj. compact K. Then

$$(I - \lambda_K K)\, u = f$$

cannot have a unique solution. A solution exists only if $f \perp N(I - \lambda_K K)$. Thus

$$(I - \lambda_K K)\, u = f - \sum_{j=1}^{N} \langle f, \phi_j \rangle \phi_j = \hat{f}$$

where $E_{\mu_K}(K) = \text{span}\{\phi_i\}$ will have a soln since

$$\hat{f} \perp N(I - \lambda_K K).$$

In this case

$$u(x) = \hat{f}(x) + \lambda_K \sum_{i=1}^{\infty} \frac{\mu_i \phi_i(x) \langle \hat{f}, \phi_i \rangle}{1 - \lambda_K \mu_i}$$

reduces to

$$u(x) = f(x) + \sum_{\mu_j \neq \mu_K} \frac{\mu_j \phi_j(x) \langle f, \phi_j \rangle}{\mu_K - \mu_j} - \sum_{\mu_j = \mu_K} \phi_j(x) \langle f, \phi_j \rangle$$

$$u(x) = f(x) + \lambda_K \hat{R} f$$

where $\hat{R}$ called the pseudo resolvent.

Degenerate compact K

dim $R(K) < \infty \Rightarrow R(K)$ closed. Fred alternativ

$$H = R(K) \oplus N(K^*)$$

if K also self adjoint

$$H = R(K) \oplus N(K)$$

Since K compact self adjoint

$R(K) = \text{span}\{\phi_i\}$ $\quad K\phi_i = \mu_i\phi_i$, $\mu_i \neq 0$ $\quad i = 1, 2 \dots n$

$N(K) = \text{span}\{\phi_i\}$ $\quad K\phi_i = 0$ $\quad$, $i = n+1, n+2, \dots$

so $\{\phi_i\}$ is a basis for H.

Now want to solve

$$(1) \qquad (I - \lambda K)u = f$$

If we let

$$u = \sum u_i\phi_i \qquad f = \sum f_i\phi_i$$

and substitute into (1)

$$\sum_{i\geq 1}(I - \lambda K)u_i\phi_i = \sum_{i\geq 1} f_i\phi_i$$

$$\sum_{i=1}^{n}(1-\lambda\mu_i)u_i\phi_i + \sum_{i>n} u_i\phi_i = \sum_{i\geq 1} f_i\phi_i$$

For $i \leq n$ find $u_i = f_i/(1-\lambda\mu_i)$. For $i > n$, $u_i = f_i$