INTEGRAL EQUATIONS - Classification

FREDHOLM INTEGRAL EQN (ϕ = unknown)

$$\int_a^b K(x,y)\phi(y)\,dy = f(x) \qquad \text{(1rst kind)}$$

$$\phi(x) - \lambda\int_a^b K(x,y)\phi(y)\,dy = f(x) \qquad \text{(2nd kind)}$$

VOLTERRA INTEGRAL EQN (ϕ = unknown)

$$\int_a^x K(x,y)\phi(y)\,dy = f(x) \qquad \text{(1rst kind)}$$

$$\phi(x) - \lambda\int_a^x K(x,y)\phi(y)\,dy = f(x) \qquad \text{(2nd kind)}$$

LINEAR INTEGRAL EQNS (Ex)

$$K(x,\phi) \equiv K_1(\phi(x)) + \int_a^b K_2(x,y,\phi(y))\,dy$$

$$K(x,\phi_1+\phi_2) = K(x,\phi_1) + K(x,\phi_2)$$

$$K(x,\alpha\phi_1) = \alpha K(x,\phi_1)$$

CONNECTION BETWEEN FREDHOLM/VOLTERRA INT. EQNS

FOR VOLTERRA, $K(x,y) = 0$ for $y > x$

SOLVED FREDHOLM INTEGRAL EQUATION

$$u(x) - \lambda \int_0^1 e^{x-y} u(y)\, dy = f(x) \qquad (1)$$

unknown here is $u(x)$.

Soln 1 Define $\eta = \int_0^1 e^{-y} u(y)\, dy$ then (1) becomes

$$u(x) - \lambda e^x \eta = f(x)$$

$$u(x) = f(x) + \eta \lambda e^x \qquad (2)$$

where η is unknown. Using (2) in defn of η we find

$$\eta = \frac{1}{1-\lambda} \int_0^1 e^{-y} f(y)\, dy \quad , \qquad \lambda \neq 1$$

Thus,

$$u(x) = f(x) + \frac{\lambda}{1-\lambda} \int_0^1 e^{x-y} f(y)\, dy \qquad \square$$

Remark: Defining the operator $Ku = \int_0^1 e^{x-y} u(y)\, dy$

$$(I - \lambda K) u = f$$

Then the solution is

$$u = \frac{1}{1-\lambda} (I + \lambda K) f$$

Remark: when $\lambda = 1$, note that $v(x) = Ae^x$ has

$$v(x) - \int_0^1 Ae^x\, dy = 0 \qquad \forall A \in \mathbb{R}$$

so that $u(x) = Ae^x + f(x)$ is a (non unique) soln family.

SOLVED VOLTERA EQUATION VIA LAPLACE TRANSFORM

$$(1) \qquad \boxed{u(t) = 1 + \int_0^t (t-\tau)\, u(\tau)\, d\tau} \qquad t>0$$

Has the form

$$u(t) = f(t) + \int_0^t k(t-\tau)\, u(\tau)\, d\tau$$

Recall facts from Laplace transform theory

$$F(s) = \mathcal{L}(f)(s) \equiv \int_0^\infty e^{-st} f(t)\, dt$$

Given convolution defn we have

$$(f*g)(t) \equiv \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

$$\mathcal{L}(f*g)(s) = F(s)\, G(s) = \mathcal{L}(f)\cdot\mathcal{L}(g)$$

Taking the $\mathcal{L}$-transform of (1), $U(s) \equiv \mathcal{L}(u)(s)$

$$\mathcal{L}(u) = \mathcal{L}(1) + \mathcal{L}(t)\,\mathcal{L}(u)$$

$$U(s) = \frac{1}{s} + \frac{1}{s^2}\, U(s)$$

$$\boxed{U(s) = \frac{s}{s^2-1}}$$ Transform of soln.

Use Inversion formula (or other means)

$$u(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, U(s)\, ds = \sum_{s=\pm 1} \mathrm{Res}\, \tilde{f}(s,t) = \cosh t$$

REMARK General coupled linear eqns can be solved same way.

Some simple conversions to Fredholm integral eqns

Claim

$$\boxed{\begin{array}{l}\text{Solution of} \\ u''(x) = f(x) \qquad u(0)=u(1)=0 \\ u(x) = \displaystyle\int_0^1 K(x,y)\, f(y)\, dy \\ K(x,y) = \begin{cases} y(x-1) & 0 \le y < x \le 1 \\ x(y-1) & 0 \le x < y \le 1 \end{cases}\end{array}}$$

First note $k(0,y) = k(1,y) = 0$ so that $u(0)=u(1)=0$.

$$u(x) = \int_0^x y(x-1) f(y)\,dy + \int_x^1 x(y-1) f(y)\,dy$$

$$u'(x) = \int_0^x y f(y)\,dy + \int_x^1 (y-1) f(y)\,dy$$

$$u''(x) = x f(x) - (x-1) f(x)$$

$$u''(x) = f(x) \qquad \square$$

EXAMPLE Convert to a Fredholm integral equation.

$$u'' + \lambda u = f(x) \qquad u(0)=u(1)=0$$

Set $F(x) = f(x) - \lambda u(x)$ and apply claim

$$u(x) + \lambda \int_0^1 k(x,y)\, u(y)\, dy = g(x)$$

where

$$g(x) = \int_0^1 k(x,y)\, f(y)\, dy$$

(Comment on nonuniqueness if $\lambda_n = n\pi$)

EXAMPLE

$$\boxed{\begin{array}{ll} u'' + a(x)u' + b(x)u = f(x) & x \in (0,1) \\ u(0) = u(1) = 0 & \end{array}}$$

Set

(1) $\quad u(x) = \phi(x)\psi(x) \qquad \psi(x) = \exp\left(-\frac{1}{2}\int^{x} a(s)\,ds\right)$

to eliminate first derivatives. Since $\psi > 0$ must have

(2) $\quad \phi(0) = \phi(1) = 0$

Substitute (1) into ODE yields

(3) $\quad \phi'' + q(x)\phi = \psi^{-1} f$

(4) $\quad q(x) = b(x) - \frac{1}{4}a^2(x) - \frac{1}{2}a'(x)$

Then, using claim

(5) $\quad \phi(x) + \int_0^1 \hat{k}(x,y)\phi(y)\,dy = \hat{f}(x)$

where

(6) $\quad \hat{f}(x) = \int_0^1 k(x,y)\psi^{-1}(y) f(y)\,dy$

(7) $\quad \hat{k}(x,y) = q(y)\,k(x,y)$

Conversion of IVP to Volterra Eqns (Linear eqn case)

(1) $L[u] = u^{(n)}(x) + a_1(x)u^{(n-1)}(x) + \cdots a_n(x)u = F(x)$

(2) $u(0) = c_0 \quad u'(0) = c_1, \ldots \quad u^{(n-1)}(0) = c_{n-1}$

Define

$$D^n u = \frac{d^n u}{dx^n} = \phi(x)$$

and for any $\psi(x)$

(3) $$D^{-1}\psi \equiv \int_0^x \psi(y)\,dy$$

Using (Laplace transforms or) integration by parts

(4) $$D^{-n}\psi = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1}\psi(y)\,dy$$

Proof for $n=2$

$$D^{-2}\psi = D^{-1}(D^{-1}\psi) = \int_0^x \underbrace{\left(\int_0^s \phi(y)\,dy\right)}_{u(s)} \underbrace{ds}_{dv(s)}$$

$$= s\int_0^s \phi(y)\,dy\Big|_{s=0}^{s=x} - \int_0^x s\,\phi(s)\,ds$$

$$= x\int_0^x \phi(y)\,dy - \int_0^x s\,\phi(s)\,ds$$

$$= \int_0^x (x-y)\phi(y)\,dy \qquad \square$$

(See back for convolution method?)

Now define the n-term Taylor series about $x=0$ of u

$$\mathcal{U}(x) = \sum_{K=0}^{n} \frac{u^{(K)}(0)}{K!} x^K \qquad \text{where} \qquad u^{(0)}(x) \equiv u(x)$$

Using initial conditions

$$\mathcal{U}(x) = \sum_{K=0}^{n} \frac{c_K}{K!} x^K$$

Now observe

$$(5) \begin{cases} \dfrac{d^{n-1}u}{dx^{n-1}} = c_{n-1} + D^{-1}\phi & = \mathcal{U}^{(n-1)}(x) + D^{-1}\phi \\ \dfrac{d^{n-2}u}{dx^{n-2}} = c_{n-1}x + c_{n-2} + D^{-2}\phi & = \mathcal{U}^{(n-2)}(x) + D^{-2}\phi \\ \quad\vdots \qquad\qquad \vdots & \qquad \vdots \\ u = \dfrac{c_{n-1}}{(n-1)!} x^{n-1} + \cdots c_1 x + c_0 + D^{-n} & = \mathcal{U}(x) + D^{-n}\phi \end{cases}$$

Using (5), (4) in (1) we find (1) can be written

$$\phi(x) + \int_0^x K(x,y)\,\phi(y)\,dy = f(x) \qquad , \quad \phi = \frac{d^n u}{dx^n}$$

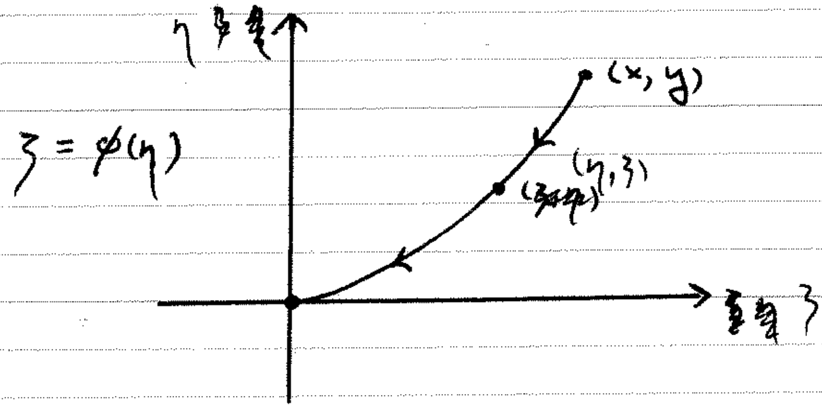
where

$$K(x,y) = \sum_{j=1}^{n} a_j(x) \frac{(x-y)^{j-1}}{(j-1)!}$$

$$f(x) = F(x) - \sum_{j=1}^{n} a_j(x)\,\mathcal{U}^{(n-j)}(x)$$

ABEL'S EQUATION

A particle of mass m initially at rest at (x,y) moves on curve $\zeta = \phi(\eta)$ in the (ζ,η)-plane through the origin. If $T(y)$ is the time taken to traverse the path, what is $\phi(\eta)$?



$\phi(0) = 0$

$\phi(y) = x$

Total energy is $E=0$. Conservation of energy

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 - mg(y-\eta) = 0$$

where s = arclength, $\frac{ds}{dt}$ = speed, implies

$$(1) \qquad \frac{ds}{dt} = \sqrt{2g(y-\eta)}$$

Basic calculus

$$(2) \qquad ds = \sqrt{1+(\phi'(\eta))^2}\, d\eta$$

Combining (1)-(2) gives

$$\frac{\sqrt{1+(\phi'(\eta))^2}\,d\eta}{\sqrt{2g(y-\eta)}} = dt$$

Since traversal time $T(y)$ is known

$$\int_0^y \frac{\sqrt{1+(\phi'(\eta))^2}}{\sqrt{2g(y-\eta)}}\,d\eta = T(y)$$

Defining $\psi(\eta) = (2g)^{-1/2}\left(1+(\phi'(\eta))^2\right)^{1/2}$ we get

$$\boxed{\int_0^y \frac{\psi(\eta)\,d\eta}{\sqrt{y-\eta}} = T(y)}$$

Abel's Integral Equation.

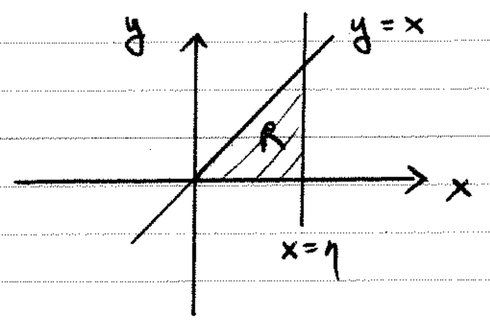
Knowing the solution $\psi(\eta)$, one can solve for $\phi(\eta)$ given initial condition $\phi(0)=0$.

SOLUTION OF ABEL'S INTEGRAL EQN. (METHOD ONE)

$$T(x) = \int_0^x \frac{u(y)\,dy}{\sqrt{x-y}}$$

Employ a trick. Multiply by $\frac{dx}{\sqrt{\eta - x}}$ and integrate from $x = 0$ to $x = \eta$

$$\int_0^\eta \frac{T(x)\,dx}{\sqrt{\eta - x}} = \int_0^\eta \left(\int_0^x \frac{u(y)}{\sqrt{\eta - x}\,\sqrt{x-y}}\, dy \right) dx$$



Integration region of RHS

$$\begin{pmatrix} y < x < \eta \\ 0 < y < \eta \end{pmatrix}$$

Interchange order of limits

$$\int_0^\eta \frac{T(x)\,dx}{\sqrt{\eta - x}} = \int_0^\eta u(y) \underbrace{\left(\int_y^\eta \frac{dx}{\sqrt{\eta - x}\,\sqrt{x-y}} \right)}_{\text{Direct integration: equals } \pi} dy$$

$$\int_0^\eta \frac{T(x)\,dx}{\sqrt{\eta - x}} = \pi \int_0^\eta u(y)\,dy$$

Differentiating this result

$$\boxed{u(\eta) = \frac{1}{\pi} \frac{d}{d\eta} \int_0^\eta \frac{T(x)}{\sqrt{\eta - x}}\, dx}$$

<u>REMARK:</u> Laplace Transforms + Convolution gives another technique

Solutions for separable kernels

$$Ku = \int_a^b \Big(\underbrace{\sum_{i=1}^{n} \phi_i(x)\,\psi_i(y)}_{k(x,y)} \Big)\, u(y)\, dy$$

Int eqn: Fredholm 2nd kind

$$u(x) = f(x) + \lambda (Ku)(x)$$

Define $\alpha_j = \langle \psi_j , u \rangle$ then ($L^2[a,b]$ inn. prod)

$$u(x) = f(x) + \lambda \sum_{i=1}^{n} \alpha_i \phi_i(x) \qquad (1)$$

$$\langle \psi_j, u \rangle = \langle \psi_j, f \rangle + \lambda \sum_{i=1}^{n} \alpha_i \langle \psi_j, \phi_i \rangle$$

$$\alpha_j = \langle \psi_j, f \rangle + \lambda \sum_{i=1}^{n} \langle \psi_j, \phi_i \rangle \alpha_i$$

Let $\alpha = (\alpha_1, \ldots \alpha_n)^T$, $\hat{f} = (\langle \psi_1, f \rangle, \ldots \langle \psi_n, f \rangle)$ and $A \in \mathbb{R}^{n \times n}$, $[A]_{ij} = \langle \psi_j, \phi_i \rangle$ gives

$$\alpha = \hat{f} + \lambda A \alpha$$

$$(I - \lambda A)\, \alpha = \hat{f}$$

If $(I - \lambda A)^{-1}$ exists, (1) gives the soln with $\alpha = (I - \lambda A)^{-1} \hat{f}$.

$$\boxed{\text{Sep Int Eqns} \iff \text{Matrix Eqns}}$$

EXAMPLE

$$u(x) = 1 + \lambda \int_0^1 (xy^2 + x^2 y)\, u(y)\, dy \qquad (1)$$

where $f(x) = 1$,

$$k(x,y) = xy^2 + x^2 y = \phi_1(x)\psi_1(y) + \phi_2(x)\psi_2(y)$$

Define

$$\alpha_1 = \langle \psi_1, u \rangle = \int_0^1 y^2 u(y)\, dy$$

$$\alpha_2 = \langle \psi_2, u \rangle = \int_0^1 y\, u(y)\, dy$$

then (1) is

$$u(x) = 1 + \lambda(\alpha_1 \phi_1(x) + \alpha_2 \phi_2(x)) \qquad (2)$$

Define

$$A = \begin{bmatrix} \langle \psi_1, \phi_1 \rangle & \langle \psi_1, \phi_2 \rangle \\ \langle \psi_2, \phi_1 \rangle & \langle \psi_2, \phi_2 \rangle \end{bmatrix} = \begin{bmatrix} 1/4 & 1/5 \\ 1/3 & 1/4 \end{bmatrix}$$

$$F = \begin{bmatrix} \langle f, \psi_1 \rangle \\ \langle f, \psi_2 \rangle \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix}$$

Then $\langle \psi_j, \cdot \rangle$ on (2) gives

$$(I - \lambda A)\,\alpha = F \qquad (3)$$

where

$$\det(I - \lambda A) = 1 - \tfrac{1}{2}\lambda - \tfrac{1}{240}\lambda^2$$

Solving (3) for $\alpha = (\alpha_1, \alpha_2)^T$ and substituting into (2)

$$u(x) = 1 + \frac{2}{3} \quad \frac{\lambda x[5x(\lambda - 36) - 6(\lambda + 20)]}{\lambda^2 + 120\lambda - 240}$$

is the solution

Eigenvalue problem

$$u(x) = \lambda \int_0^1 k(x,y)\, u(y)\, dy = \lambda \mathbb{K} u$$

re-written

$$\mathbb{K} u = \tfrac{1}{\lambda} u$$

thus $\frac{1}{\lambda}$ is e-value of $\mathbb{K}$. For $f = 0$ we find e-val eqn is

$$\det(I - \lambda A) = 0$$

$$1 - \tfrac{1}{2}\lambda - \tfrac{1}{240}\lambda^2 = 0$$

$$\lambda_{\pm} = -60 \pm 16\sqrt{15}$$

Eigenfns

$$u(x) = \alpha_{\pm}^T (\phi_1(x), \phi_2(x)) \qquad \alpha_{\pm} \in N(I - \lambda_{\pm} A)$$

EIGENVALUE EXAMPLE

Let $D = \{(x,y): x^2+y^2 \le 1\}$ and define

$$Ku = \iint_D (x^2+y^2)\, u(x,y)\, dA.$$

Compute eigenvalue and normalized e-fn. $(L^2(D))$

$$Ku = \lambda u = \alpha_1 \qquad \alpha_1 \in \mathbb{R}$$

Thus $u = \alpha_1/\lambda$ and $(\lambda \neq 0)$

$$Ku = \int_0^{2\pi}\int_0^1 r^2 \cdot \left(\frac{\alpha_1}{\lambda}\right) r\, dr\, d\theta$$

$$Ku = 2\pi \cdot \frac{1}{4} \cdot \frac{\alpha_1}{\lambda} = \alpha_1$$

From which

$$\lambda = \frac{\pi}{2}$$

Since $u = \text{const} = \bar{u}$, need

$$\|u\|^2 = \iint_D \bar{u}^2\, dA = \pi \bar{u}^2 = 1$$

So

$$(\lambda, u) = \left(\frac{\pi}{2}, \frac{1}{\sqrt{\pi}}\right).$$

For $\lambda = 0$ eval, note $\phi_n(r,\theta) \in \{\sin n\theta, \cos n\theta\}$

$$K\phi_n = 0.$$

EXAMPLE

$$Ku \equiv \int_a^b k(x,y)\, u(y)\, dy = \sum_{i=1}^{N} \phi_i(x) \int_a^b \psi_i(y)\, u(y)\, dy$$

The range $R(\underline{K}) \subset \text{span}\{\phi_i\}$ thus $\dim R(\underline{K}) = N$.

If ϕ_i independent, $N(\underline{K}) = \{0\}$ and $\lambda = 0$ not e-value.

$$\alpha_i \phi_i(x) = 0 \qquad \forall x \in (a,b)$$

Eigenvalue problem

$$Ku = \lambda u$$

$$\alpha_i \phi_i = \lambda u$$

$$(A - \lambda I)\alpha = 0$$

Thus

$$\det(A - \lambda I) = 0 \qquad \text{e-value eqn}$$

$$u = \frac{1}{\lambda} \sum \alpha_i \phi_i(x) \qquad \text{e-fns}$$

where $\alpha \in N(A - \lambda I)$.

Systems of integral eqns (Fred 2nd kind)

$$\vec{u} + \lambda K \vec{u} = \vec{f}$$

where $\vec{u} = (u_1, \dots u_n)$, $\vec{f} = (f_1, \dots f_n)$ and

$$[K\vec{u}]_i = \int_a^b \sum_{j=1}^n k_{ij}(x,y)\, u(y)\, dy.$$

Solution via complete sets

Let $\{\phi_n\}$ be a complete orthonormal set for $L^2[a,b]$.

$$u - \lambda K u = f$$

Let $u = \sum_{i=1}^n u_i \phi_i(x)$, $f = \sum_{i=1}^n f_i \phi_i(x)$. Set

$$(\mathrm{I}) \qquad Ku = \sum_{i=1}^n \alpha_i K\phi_i$$

$$\sum (\phi_i - \lambda K\phi_i)\, u_i = \sum f_i \phi_i$$

$$\langle \phi_j, \phi_j - \lambda K\phi_j \rangle u_j = f_j$$

determines u_j for solution. But

1) How do we know (I) is valid?

2) If $1/\lambda$ is an eval of K then some u_j (how many) cannot be found.

Goal of Chapter has to do with

$$(1) \qquad u(x) + \int_a^b K(x,y)\, u(y)\, dy = f(x)$$

Suppose

$$Ku \equiv \int_a^b k(x,y)\, u(y)\, dy \quad , \quad K : L^2[a,b] \to L^2[a,b]$$

generates a basis for $H = L^2[a,b]$ via

$$K\phi_n = \lambda_n \phi_n \quad , \quad \|\phi_n\| = 1$$

Then

$$u(x) = \sum_{n \geq 1} u_n \phi_n(x) \qquad f(x) = \sum_{n \geq 1} f_n \phi_n(x)$$

in (1) gives

$$\sum u_n \phi_n + \sum u_n K\phi_n = \sum f_n \phi_n$$

$$\sum [(1-\lambda_n)u_n - f_n]\phi_n = 0$$

Thus

$$u_n = \frac{f_n}{1-\lambda_n} \quad , \quad \lambda_n \neq 1$$

gives soln.